[**DRAFT**] A Peripatetic Course in Algebraic Topology

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Abstract

These notes are based on lectures in algebraic topology taught by Peter May and Henry Chan at the 2016 University of Chicago Math REU. They are loosely chronological, having been reorganized for my benefit and significantly annotated by my personal exposition, plus solutions to in-class/HW exercises, plus content from readings (from May's *Finite Book*), books (e.g. May's *Concise Course*, Munkres' *Elements of Algebraic Topology*, and Hatcher's *Algebraic Topology*), Wikipedia, etc.

Part I
Foundations + Weeks 1 to 3

1 Topological notions

Remark. This is a review of material from point-set topology. See Munkres, or the readings [1.1-1.5] (from which some of the definitions are directly taken).

1.1 Topological spaces

Definition 1.1. A **topological space** $(X, \mathcal{U})$ is a set X, along with a set of subsets $\mathcal{U}$ of X we call a **topology** on X. One requires that $\emptyset, X \in \mathcal{U}$, and that $\mathcal{U}$ is closed under finite intersection and arbitrary union. If the topology is apparent, one simply uses X to refer to the space itself.

Definition 1.2. The elements of $\mathcal{U}$ are called the **open sets** of X (in the topology $\mathcal{U}$). The complements of open sets in X are called **closed sets**. A **neighborhood** of a point $x \in X$ is an open set U such that $x \in U$.

Definition 1.3. The **closure** of a subset $A \subseteq X$, denoted $\overline{A}$, is the intersection of all closed sets containing A. Equivalently, $\overline{A}$ is the union of A and its **limit points** (points x for which every neighborhood intersects A at some $a \neq x$).

Definition 1.4. A **basis** for a topology on X is a set $\mathcal{B}$ of subsets such that every $x \in X$ is in some $B \in \mathcal{B}$, and if $x \in B' \cap B''$ then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B' \cap B''$. (Hence, every finite intersection in $\mathcal{B}$ is the union of these extant $B \in \mathcal{B}$ as well.) In this way, the unions of the sets in $\mathcal{B}$ define **the topology $\mathcal{U}$ generated by basis** $\mathcal{B}$ on X. Equivalently, we say that $\mathcal{B}$ is a **basis for the topology** $\mathcal{U}$.

Definition 1.5. A **subbasis** for a topology on X is a cover $\mathcal{S}$ of X. By including the finite intersections of $\mathcal{S}$, we get a basis $\mathcal{B}$ on X. This basis generates a topology $\mathcal{U}$ on X, and we say that $\mathcal{S}$ is a **subbasis for the topology** $\mathcal{U}$.

Remark. Observe that a subbasis $\mathcal{S}$ of a topology $\mathcal{U}$ is a weaker version of a basis $\mathcal{B}$. When a subbasis is available, however, it is a efficient description of a topology (based on the number of sets we have to describe; by the measure of simplicity of operations, a basis is more "efficient" because we only have to consider unions). In both the basis and subbasis cases, we can think of the topology $\mathcal{U}$ as the smallest topology containing $\mathcal{S}$ or $\mathcal{B}$.

1.2 Separation properties

Definition 1.6. The **separation properties** are the following hierarchy of types of a topological space:

- A T_0**-space** is one where "points are distinguished", i.e., for every two points, one of the points has an open neighborhood not containing the other.
- A T_1**-space** is one where each point is a closed subset.
- A T_2**-space** or **Hausdorff space** if any two points have disjoint open neighborhoods.

Proposition 1.7. *T_2 implies T_1 implies T_0.*

Proof. The first implication follows from fixing your desired point x, and taking the union of all the open neighborhoods disjoint from it. By the Hausdorff property, that union includes all points except x. The second implication follows from noting the complement of a point is the desired open neighborhood distinguishing it from the other point. □

Proposition 1.8. *Metric spaces are Hausdorff.*

Proof. Two distinct points x, y have a non-zero distance ϵ, which means they are separated by the disjoint basis elements $B(x; \frac{\epsilon}{2})$, $B(y; \frac{\epsilon}{2})$. □

1.3 Continuity and operations on spaces

Definition 1.9. A **continuous map** of topological spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ is a function $f : X \to Y$ such that $f^{-1}(V) \in \mathcal{U}$ for all $V \in \mathcal{V}$. That is, the preimages of open sets are open.

Remark. It suffices to check that $f^{-1}(V)$ is open for each V in a basis or a subbasis of $\mathcal{V}$, as preimages preserve union and intersection.

Definition 1.10. X and Y are **homeomorphic** if there exists a continuous map $f : X \to Y$ (called a **homeomorphism**) with a continuous inverse.

Remark. Homeomorphisms play the role of isomorphisms for topological spaces. Two spaces that are isomorphic (e.g., (a, b) and $\mathbb{R}$) have the same topological properties. Intuitively, a homeomorphism is a bijection on the underlying set, and a bijection on the open sets that preserves the relations between the open sets in the topology.

Definition 1.11. The **subspace topology** on $A \subseteq X$ is given by $\{A \cap U \mid U \text{ open in } X\}$. More abstractly, let $i : A \to X$ be an injection. Then the subset topology is given by the set of $i^{-1}(U)$ for open U.

Example 1.12. Consider $[0, 1] \subseteq \mathbb{R}$ where $\mathbb{R}$ has the usual Euclidean topology. The open sets of $[0, 1]$ with the subspace topology are the usual open sets, except truncated to $[0, \dots)$ and $(\dots, 1]$ when they cross these boundaries.

Definition 1.13. Let $\sim$ be an equivalence relation on $(X, \mathcal{U})$. The **quotient topology** is given on the set of equivalence classes $X/\sim$ where the open sets are sets of equivalence classes whose unions were open in X. That is, the topology $\mathcal{U}'$ on $X/\sim$ is

$$\mathcal{U}' = \left\{ U \subseteq X/\sim \mid \bigcup_{[a]\in U} [a] \in \mathcal{U} \right\}.$$

Equivalently, let $q : X \to Y$ be a surjection. Then we write $Y = X/\sim$ and define its open sets as $U \subseteq Y$ such that $q^{-1}(U)$ is open in X.

Example 1.14. The quotient topology enables the general method of **gluing**, whereby points in a space are identified together to give a new space. Consider $[0,1] \subseteq \mathbb{R}$ with a subspace topology, and take the equivalence relation given by the partition:

$$\sim = \{\{0,1\}\} \sqcup \{\{x\} \mid x \in (0,1)\}$$

One can envision this as gluing the ends of the interval together, and so we expect to get the space of a circle (with, e.g., its subspace topology in $\mathbb{R}^2$). One can check that $f(x) = e^{2i\pi x}$ is well-defined on $[0,1]/\sim$ and a homeomorphism to S^1.

Remark. In a sense made precise later, the definitions for subspace and quotient topologies are dual to each other (they are with respect to an injection and a surjection, respectively).

Definition 1.15. The **disjoint union topology** on $X \sqcup Y$ takes as a basis the disjoint unions of an open set of X and an open set of Y (though they are in fact all the open sets).

Example 1.16. For the *disjoint union topology*, think about $\mathbb{R}^1 \sqcup \mathbb{R}^1$. The two spaces are essentially independent (though your set has to be open in both to be called open in the union topology).

Definition 1.17. The **product topology** on $X \times Y$ takes as a *basis* the products $U \times V$ where U is open in X and V is open in Y. In the case of infinite products $\prod X_i$, *we only allow basis elements whose components are X_i in all but finitely many places.*

Example 1.18. For the *product topology*, think about $\mathbb{R}^2$'s metric space topology, which is equivalent to the product topology. The open sets are not just the (boundaryless) unions of rectangles that are products of $\mathbb{R}^1$'s open sets; these merely serve as a basis. The open sets are arbitrary unions of (arbitrarily small) rectangles; cf., the open disc.

2 Algebraic notions

Remark. This section should be review, hence the terseness. Based on class, Dummit and Foote's *Abstract Algebra*, and http://www.math.umn.edu/~garrett/m/algebra/notes/27.pdf.

2.1 Rings and modules

Definition 2.1. A **(commutative, unital) ring** R is a set equipped with two binary operations $+$ and $\cdot$ such that R is an abelian group under addition, an abelian **monoid** (think group without the inverse requirement) under multiplication, and that **distributivity** holds:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

Example 2.2. We have the rings $\mathbb{Z}$, $\mathbb{Q}$, the polynomial rings $R[x]$ for any ring R, the continuous functions $f : \mathbb{R} \to \mathbb{R}$, etc., where $+$ and $\cdot$ correspond to addition and multiplication in the usual way.

Definition 2.3. An R-**module** M is an abelian group M with an "R-action" (i.e., an action with respect to both R's additive group and R's multiplicative monoid)

$$\cdot : R \times M \to M.$$

Explicitly, we must have for all $x, y \in M$ and $r, s \in R$:

$$\begin{aligned} r \cdot (x + y) &= r \cdot x + r \cdot y \\ (r + s) \cdot x &= r \cdot x + s \cdot x \\ (rs) \cdot x &= r \cdot (s \cdot x) \\ 1_R \cdot x &= x, \end{aligned}$$

where 1_R denotes R's multiplicative identity.

Definition 2.4. The group of R-**module homomorphisms** $\mathrm{Hom}_R(M, N)$ (also known as R-**linear maps**) is given by the set of maps $f : M \to N$ satisfying for all $x, y \in M$ and $r \in R$:

$$\begin{aligned} f(x + y) &= f(x) + f(y) \\ f(rx) &= rf(x) \end{aligned}$$

These maps "preserve" the structure of the module. More formally, we are demanding that f is a group homomorphism that is **equivariant** (with respect to R's multiplicative action). One can verify that $\mathrm{Hom}_R(M, N)$ is a group by pointwise addition.

Example 2.5. For $R = k$ a field, the concept of R-module coincides exactly with being a k-vector space.

Example 2.6. For $R = \mathbb{Z}$, the concept of R-module coincides exactly with being an abelian group (the action is given by repeated addition: $ng = g + \cdots + g$).

Example 2.7. Polynomial rings $R[x]$ are R-modules.

Definition 2.8. A **submodule** N of an R-module M is an abelian subgroup of M that is also closed under multiplication by R.

Example 2.9. For $N \subseteq M = R$, then N is a submodule exactly when it is an ideal of R.

Definition 2.10. The **quotient module** M/N given by an R-module M and a submodule N is the abelian subgroup M/N with the induced R-action $r(m+N) = rm+N$. We verify this is well-defined: if $m+N = m'+N$, then $m-m' \in N$. Then $r(m-m') = rm-rm' \in N$ since N is a submodule. Then $rm+N = rm'+N$ as desired.

2.2 Tensor products

The motivation for the tensor product $M \otimes_R N$ of modules M, N is that it is the "minimal" setting from which bilinear maps $M \times N \to P$ can be studied as linear maps (i.e., R-module homomorphisms) $M \otimes_R N \to P$. For example, if there is a natural notion of multiplication between M and N, that is a bilinear map and is thus ideally studied in $M \otimes_R N$.

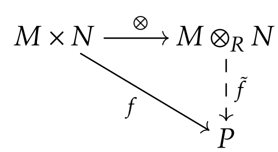
Definition 2.11. Let $f : M \times N \to P$ be an R-**bilinear** module homomorphism, i.e.,

$$\begin{aligned} f(m+m',n) &= f(m,n)+f(m',n) \\ f(rm,n) &= rf(m,n) \\ f(m,n+n') &= f(m,n)+f(m,n') \\ f(m,rn) &= rf(m,n) \end{aligned}$$

hold. The **tensor product** $M \otimes_R N$ is uniquely defined (up to unique isomorphism) as an R-module equipped with an R-bilinear map

$$\otimes : M \times N \to M \otimes_R N$$

such that any such R-bilinear map f, there exists a unique R-linear (i.e., an R-module homomorphism) map $\tilde{f}$ such that



commutes; that is, $f = \tilde{f} \circ \otimes$.

Remark. By **uniquely defined up to unique isomorphism**, we mean that for two tensor products T, T' and associated $\otimes, \otimes'$, one can prove there is a unique isomorphism $i : T \cong T'$ which satisfies $\otimes' = i \circ \otimes$. This is why we can safely write $M \otimes_R N$. This characteristic is typical of **universal properties**, of this definition is an example.

However, this does not guarantee that a tensor product actually exists; it just states that if one did exist, it would be unique in this manner. The following rectifies that:

Proposition 2.12. *Given $M \times N$, there exists a tensor product $M \otimes_R N$.*

$$M \otimes_R N = F(M \times N)/L$$

where $F(M \times N)$ denotes the free module over the set $M \times N$, and L is the submodule generated by

$$\begin{aligned} (m+m', n) - ((m,n) + (m', n)) \\ (m, n+n') - ((m,n) + (m,n')) \\ c(m,n) - (cm, n) \\ c(m,n) - (m, cn) \end{aligned}$$

(implicitly working in the image of $\iota : M \times N \to F(M \times N)$).

Proof. See Garrett's notes (the universal property of a free module also comes into play). The intuitive idea is that the submodule L encodes the exact relations that enforce bilinearity on the "obvious" inclusion ι. □

Example 2.13. Let V, W be vector spaces over F. They have bases $\{v_i\}, \{w_j\}$. The map $\otimes$ is given by the bilinear map on the basis $(v_i, w_j) \to v_i \otimes w_j$, *then extended by linearity*. Hence the induced $\tilde{f}$ (see the universal property) must satisfy

$$\tilde{f}(v_i \otimes w_j) = f(v_i, w_j).$$

Example 2.14. By the tensor product relations, we have in $M \otimes N$:

$$m \otimes 0 = m \otimes (0+0) = (m \otimes 0) + (m \otimes 0),$$

which implies $m \otimes 0 = 0$ for all $m \in M$. The same reasoning gives $0 \otimes n = 0$ for all $n \in N$.

Example 2.15. Note that

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(3) \cong 0.$$

To see this, note that $3a = a$ for $a \in \mathbb{Z}/(2)$ but $3b = 0$ for $b \in \mathbb{Z}/(3)$. $3(a,b) = (a,b) = (a,0)$. Meanwhile, $4(a,b) = (a,b) = (0,b)$. Hence $(a,b) = (a,0) = (0,b) = (0,0)$.

Exercise 2.16. *Let I, J be ideals of R. Show that*

$$R/I \otimes_R R/J \cong R/(I+J).$$

Proof. Observe that

$$\phi : R/I \times R/J \to R/(I+J), \quad \phi(a+I, b+J) = (a+I)(b+J) = ab + (I+J)$$

is a bilinear map. For example,

$$\begin{aligned}\phi(a+a'+I, b+J) &= (a+a'+I)(b+J) = (a+a')b + (I+J) \\ &= \phi(a+I, b+J) + \phi(a'+I, b+J).\end{aligned}$$

Hence, it factors through a linear map

$$\tilde{\phi} : R/I \otimes_R R/J \to R/(I+J)$$

of R-modules. Certainly $\tilde{\phi}$ is surjective, since we now know the following is well-defined:

$$\tilde{\phi}(m+I \otimes 1+J) = (m+I)(1+J) = m + (I+J).$$

To show injectivity, we first note that every element in $R/I \otimes_R R/J$ can be a written as a simple tensor of a particular form. Namely, we can rewrite a general tensor as:

$$\begin{aligned}\sum_i (m_i + I \otimes n_i + J) &= \sum_i m_i n_i (1 + I \otimes 1 + J) = \left(\sum_i m_i n_i\right)(1 + I \otimes 1 + J) \\ &= \left(\left(\sum_i m_i n_i\right) + I \otimes 1 + J\right).\end{aligned}$$

Then every element of $R/I \otimes_R R/J$ is of the form $m + I \otimes 1 + J$. Then if

$$0 + (I+J) = \tilde{\phi}(m + I \otimes 1 + J) = m + (I+J)$$

implies $m \in I + J$. Hence we can write $m = a + b$ where $a \in I$ and $b \in J$ and get

$$\begin{aligned}m + I \otimes 1 + J &= a + b + I \otimes 1 + J = b + I \otimes 1 + J = b(1 + I \otimes 1 + J) = 1 + I \otimes b + J = 1 + I \otimes J \\ &= 0,\end{aligned}$$

which shows the kernel is trivial. □

Example 2.17. We have

$$\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n) \cong \mathbb{Z}/((m)+(n)) = \mathbb{Z}/(\gcd(m,n)).$$

We also have

$$k[x]/(f) \otimes_{k[x]} k[x]/(g) \cong k[x]/((f)+(g)),$$

and if k a field then the right side is equal to $k[x]/(\gcd(f,g))$.

Exercise 2.18. *Verify the following properties: that R is unital with respect to $\otimes_R$, that $\otimes_R$ is associative, and that $\otimes_R$ is commutative. That is:*

$$R \otimes_R M \cong M \cong M \otimes_R R$$
$$M \otimes_R (N \otimes_R P) \cong (M \otimes_R N) \otimes_R P$$
$$M \otimes_R N \cong N \otimes_R M.$$

Proof. Let $r, r' \in R$, $m, m' \in M$, $n, n' \in N$, $p, p' \in P$. Then:

- Observe that $\phi : (r, m) \mapsto rm$ is a bilinear map:

  $$\phi(r + r', m) = (r + r')m = rm + r'm = \phi(r, m) + \phi(r', m)$$
  $$\phi(r, m + m') = r(m + m') = rm + rm' = \phi(r, m) + \phi(r, m').$$

  It thus factors through a map $\tilde{\phi} : R \otimes_R M \to M$. We define the map $\tilde{\psi}(m) = 1 \otimes_R m$. One checks these are inverses; e.g.,

  $$\tilde{\psi}(\tilde{\phi}(r \otimes_R m)) = \tilde{\psi}(rm) = 1 \otimes_R rm = r \otimes_R m.$$

- The following strategy is due to this comment. Observe that $\phi_p : (m, n) \mapsto (m, n \otimes_R p)$ is a bilinear map for fixed p:

  $$\phi(m + m', n) = (m + m', n \otimes_R p) = (m, n \otimes_R p) + (m', n \otimes_R p) = \phi(m, n) + \phi(m', n)$$
  $$\phi(m, n + n') = (m, (n + n') \otimes_R p) = (m, n \otimes_R p) + (m, n' \otimes_R p) = \phi(m, n) + \phi(m, n')$$

  It thus factors through $\tilde{\phi}_p : M \otimes_R N \to M \otimes_R (N \otimes_R P)$. Let $a, a' \in M \otimes_R N$. Then $\psi : (a, p) \mapsto \tilde{\phi}_p(a)$ is a bilinear map:

  $$\psi(a + a', p) = \tilde{\phi}_p(a + a') = \tilde{\phi}_p(a) + \tilde{\phi}_p(a') = \psi(a, p) + \psi(a', p)$$
  $$\psi(a, p + p') = \tilde{\phi}_{p+p'}(a) = \tilde{\phi}_p(a) + \tilde{\phi}_{p'}(a) = \psi(a, p) + \psi(a, p')$$

  where the equality $\tilde{\phi}_{p+p'} = \tilde{\phi}_p + \tilde{\phi}_{p'}$ can be verified by writing the general sum $a = \sum_i m_i \otimes n_i$ and using the linearity of $\tilde{\phi}_p$ on $M \otimes_R N$. Hence, ψ factors through

  $$\tilde{\psi} : (M \otimes_R N) \otimes_R P \to M \otimes_R (N \otimes_R P).$$

  One can construct an R-module homomorphism in the other direction in a similar manner, and verify that they are inverses.

- Observe that $\phi : (m, n) \mapsto n \otimes_R m$ and $\psi : (n, m) \mapsto m \otimes_R n$ are bilinear maps. For example,

  $$\phi(m + m', n) = n \otimes_R (m + m') = n \otimes_R m + n \otimes_R m' = \phi(m, n) + \phi(m', n)$$
  $$\phi(m, n + n') = (n + n') \otimes_R m = n \otimes_R m + n' \otimes_R m = \phi(m, n) + \phi(m, n').$$

They thus factor through $\tilde{\phi} : M \otimes_R N \to N \otimes_R M$ and $\tilde{\psi} : N \otimes_R M \to M \otimes_R N$ respectively. One confirms they are inverses; e.g.,

$$\tilde{\psi}(\tilde{\phi}(m \otimes_R n)) = \tilde{\psi}(n \otimes_R m) = m \otimes_R n.$$

□

Example 2.19. Consider the following sets and elements f and g:

$$(f : M \times N \to P) \in \mathrm{Hom}_R(M \times N, P)$$
$$(g : M \to \mathrm{Hom}_R(N, P)) \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)).$$

One might hope $\mathrm{Hom}_R(M \times N, P)$ and $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ are equal, e.g., via the "obvious" **currying map**

$$(g(m))(-) := f(m, -).$$

However, this is not the case; the reason is that $f(m, -)$ is not guaranteed to be an R-module homomorphism (i.e., an element of $\mathrm{Hom}_R(N, P)$)! In fact, the requirement for $f(m, -)$ to be a homomorphism (and for the $f \mapsto g$ map itself to be a homomorphism) is exactly the requirement for $f(-, -)$ to be bilinear:

$$(g(m))(r-) = f(m, r-) = rf(m, -) = r((g(m))(-))$$
$$(g(rm))(-) = f(rm, -) = rf(m, -) = (rg(m))(-).$$

Making this rigorous, one concludes that

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$$

is the correct statement. This is the **tensor-hom adjunction**, which is a special case of adjoint functors that we will revisit in the section on category theory.

3 Categorical notions

3.1 Categories

Remark. This treatment is mostly from class. http://mimosa.pntic.mec.es/jgomez53/matema/docums/villarroel-categories.pdf was used as an additional reference.

Definition 3.1. A **category** $\mathcal{C}$ consists of

- A class of **objects** $\mathrm{ob}(\mathcal{C})$.
- A class of **morphisms** $\hom_{\mathcal{C}}(X, Y)$ for each pair of objects $X, Y \in \mathrm{ob}(\mathcal{C})$. Here, X is called the **source** and Y is called the **target**.

- A **composition map** $\hom_{\mathcal{C}}(X,Y) \times \hom_{\mathcal{C}}(Y,Z) \to \hom_{\mathcal{C}}(X,Z)$ for every three objects $X,Y,Z \in \mathrm{ob}(\mathcal{C})$. The composition of f,g is written $g \circ f$.

such that the following axioms

- **Associativity:** If $f \in \hom_{\mathcal{C}}(W,X)$, $g \in \hom_{\mathcal{C}}(X,Y)$, $f \in \hom_{\mathcal{C}}(Y,Z)$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- **Identity:** For each object X there is an **identity morphism** $\mathrm{id}_X \in \hom_{\mathcal{C}}(X,X)$, such that every morphism $f \in \hom_{\mathcal{C}}(Y,Z)$ satisfies

$$f \circ \mathrm{id}_Y = f = \mathrm{id}_Z \circ f.$$

hold.

Remark. Think of categories as classes of dots and arrows between dots, where if two arrows occur in sequence, they (implicitly) define a composed arrow. Associativity holds, and every dot has an implicit identity arrow to itself.

Definition 3.2. An **endomorphism** is a morphism with the same source and target, i.e., an element of $\hom_{\mathcal{C}}(X,X)$ for some object X.

Definition 3.3. An **isomorphism** is a morphism $f \in \hom_{\mathcal{C}}(X,Y)$ if there exists an **inverse morphism** $g \in \hom_{\mathcal{C}}(Y,X)$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. The inverse morphism is necessarily unique.

Definition 3.4. A category $\mathcal{C}$ is:

- **Small** if $\mathrm{ob}(\mathcal{C})$ and all $\hom_{\mathcal{C}}(X,Y)$ are sets.
- **Large** if it is not small.
- **Locally small** if all $\hom_{\mathcal{C}}(X,Y)$ are sets.

Example 3.5. We can formulate the different types of mathematical objects we've encountered thus far as living and interacting in their respective categories. The following are all locally small categories:

	Set	**Top** (**Top**$_\star$)	**Grp** (**Ab**)	**Vect**$_{\mathbf{k}}$, **R**-**Mod**
objects	all sets	top spaces (with basepoint)	(abelian) groups	$\cdots$
morphisms	functions	(pointed) cont functions	homomorphisms	$\cdots$
isomorphisms	bijection	(pointed) homeomorphism	isomorphism	$\cdots$

Definition 3.6. The **category associated to a poset** X is the small category where the objects are elements of X, and the morphisms are

$$X(x,y) = \begin{cases} (x \leq y) & \text{if } x \leq y \\ \emptyset & \text{otherwise.} \end{cases}$$

Composition is given by $(y \leq z) \circ (x \leq y) = (x \leq z)$ and identity is given by $\mathrm{id}_x = (x \leq x)$.

Definition 3.7. The **category associated to a monoid** is the small category with one object $*$. The set of (endo)morphisms is $\hom_{\mathcal{C}}(*,*) = G$, where the composition law is given by $h \circ g = gh$.

Definition 3.8. The **category associated to a group**, denoted $\mathcal{C}_G$ for a group G, is the category associated to it as a monoid. However, in this case every morphism is an isomorphism, since group elements have inverses.

Exercise 3.9. *Conversely, show that a small category with one object such that every morphism is an isomorphism, is a group.*

Proof. Consider the set $\hom_{\mathcal{C}_G}(*,*)$. The composition map gives a closed binary operation

$$\hom_{\mathcal{C}_G}(*,*) \times \hom_{\mathcal{C}_G}(*,*) \to \hom_{\mathcal{C}_G}(*,*)$$

and the axioms ensure associativity. The morphism id_* satisfies the properties of a group identity, and since every $f \in \hom_{\mathcal{C}_G}(*,*)$ is an isomorphism, there exists g such that $f \circ g = g \circ f = \mathrm{id}_*$, which is exactly the inverse property for a group. □

Definition 3.10. A **groupoid** is a small category where every morphism is an isomorphism.

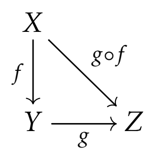
Remark. In this way, we see that monoids are essentially the same as small categories with a single object. Groups are the subclass where the single object's endomorphisms are all isomorphisms, i.e., a group is a groupoid with only one object.

3.2 Functors

Definition 3.11. A **(covariant) functor** between categories $F : \mathcal{C} \to \mathcal{D}$:

- Sends every object in $\mathcal{C}$ to an object in $\mathcal{D}$
- Sends every morphism in $\hom_{\mathcal{C}}(X,Y)$ to a morphism in $\hom_{\mathcal{D}}(F(X),F(Y))$, such that $F(g \circ f) = F(g) \circ F(f)$ and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Remark. The last property can be viewed as the requirement that commutative diagrams are preserved:



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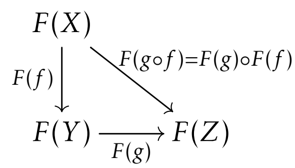


Example 3.12. Let G be a group with an action ϕ on a set X. This action can be viewed as a functor

$$F_\phi : \mathcal{C}_G \to \mathbf{Set}$$

given by $F_\phi(*) = X$, which maps morphisms from $\mathcal{C}_G(*,*)$ to $\mathbf{Set}(X,X)$. Hence the action is given by

$$\phi : G \times X \to X, \quad \phi(g,x) = (F_\phi(g))(x).$$

Similarly, $F : \mathcal{C}_G \to \mathbf{Top}$ describes a G-space, and $F : \mathcal{C}_G \to \mathbf{Vect_k}$ describes a G-representation.

Example 3.13. Taking the free group is a functor $F : \mathbf{Set} \to \mathbf{Grp}$. Set maps become group homomorphisms between free groups.

Example 3.14. A functor $\mathcal{C}_G \to \mathcal{C}_H$ between categories associated to two groups are group homomorphisms $G \to H$.

Example 3.15. A functor between the categories associated to two posets can be viewed as an order-preserving function.

Exercise 3.16. *If $F : \mathcal{C} \to \mathcal{D}$ is a functor and $f \in \hom_{\mathcal{C}}(X,Y)$ is an isomorphism, then $F(f)$ is an isomorphism.*

Proof. Let $f \in \hom_{\mathcal{C}}(X,Y)$ an isomorphism with inverse g. Then

$$F(f) \circ F(g) = F(f \circ g) = F(\mathrm{id}_{F(Y)})$$
$$F(g) \circ F(f) = F(g \circ f) = F(\mathrm{id}_{F(X)})$$

which shows $F(f)$ is an isomorphism with inverse $F(g)$. □

Remark. The contrapositive of the above is what makes **invariants** a powerful concept. For example, if the fundamental groups of two topological spaces are not isomorphic, then they cannot be homeomorphic spaces.

Definition 3.17. A **forgetful functor** is a covariant functor that loses some of the algebraic properties of the source category.

Example 3.18. There are canonical forgetful functors from each of **Grp**, **Top**, $\mathbf{Vect_k}$ to **Set** (forget the structure implied on each object's underlying set by the restricted class of morphisms). These functors are not as trivial as one might expect. Consider **Grp** $\to$ **Set**. Group homomorphisms become non-privileged subsets of the set maps. Meanwhile, multiple groups will map to their shared underlying set.

Definition 3.19. A **contravariant functor** between categories $F : \mathcal{C} \to \mathcal{D}$:

- Sends every object in $\mathcal{C}$ to an object in $\mathcal{D}$
- Sends every morphism in $\hom_{\mathcal{C}}(X, Y)$ to a morphism in $\hom_{\mathcal{D}}(F(Y), F(X))$, such that $F(g \circ f) = F(f) \circ F(g)$ and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Remark. The last property can be viewed as the requirement that commutative diagrams are preserved, but with arrows in the opposite direction:

$$\begin{array}{ccc} X & & \\ {\scriptstyle f}\downarrow & \searrow^{g\circ f} & \\ Y & \xrightarrow[g]{} & Z \end{array} \qquad \text{goes to} \qquad \begin{array}{ccc} F(X) & & \\ {\scriptstyle F(f)}\uparrow & \nwarrow^{F(g\circ f)=F(f)\circ F(g)} & \\ F(Y) & \xleftarrow[F(g)]{} & F(Z) \end{array}$$

Remark. One can still study contravariant functors in the same realm as (covariant) functors as follows: let the **opposite category** $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ be the category where the source and target of every morphism in $\mathcal{C}$ is reversed. Then contravariant functors $\mathcal{C} \to \mathcal{D}$ are exactly covariant functors $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$.

Example 3.20. A map $f : V \to W$ of k-vector spaces has a corresponding dual map that goes in the opposite direction

$$f^* : \mathrm{Hom}(W, k) \to \mathrm{Hom}(V, k).$$

That is, $\mathrm{Hom}(-, k) : \mathbf{Vect_k} \to \mathbf{Vect_k}$ reverses the direction of morphisms, making it a contravariant functor (i.e., a functor $\mathbf{Vect_k} \to \mathbf{Vect_k^{op}}$). A quick way to see this is that passing to a map's dual is equivalent to taking the transpose of its matrix, and that $(AB)^T = B^T A^T$.

3.3 Natural transformations

Definition 3.21. A **natural transformation** is a family of morphisms written $\alpha : F \implies G$, where $F, G : \mathcal{C} \to \mathcal{D}$, where each $Z \in \mathrm{ob}(\mathcal{C})$ has a morphism $\alpha_Z \in \hom_{\mathcal{D}}(F(Z), G(Z))$ such that for all $\phi \in \hom_{\mathcal{C}}(X, Y)$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\phi)} & F(Y) \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ G(X) & \xrightarrow[G(\phi)]{} & G(Y) \end{array}$$

Remark. Intuitively, natural transformations are like morphisms of functors. They transform one functor to another while respecting the associated categories. The commutative diagram expresses that every $\phi : X \to Y$ turning into $F(\phi) : F(X) \to F(Y)$, are fixed morphisms α_X, α_Y away from being $G(\phi) : G(X) \to G(Y)$.

Example 3.22. Consider the functors $\text{id}, (-)^{**} : \mathbf{Vect_k} \to \mathbf{Vect_k}$, where $(-)^{**}$ is the "double-dual functor" given by

$$V \mapsto V^{**} = (V^*)^* = \text{Hom}(V,k)^* = \text{Hom}(\text{Hom}(V,k),k),$$

and take $\alpha_V : V \to V^{**}$ given by $v \mapsto (f \mapsto f(v))$. One can verify that

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \alpha_V}\downarrow & & \downarrow{\scriptstyle \alpha_W} \\ V^{**} & \xrightarrow[f^{**}]{} & W^{**} \end{array}$$

commutes. Hence we have a natural transformation $\alpha : \text{id} \implies (-)^{**}$.

Definition 3.23. $\alpha : F \implies G$ is a **natural isomorphism** if α_C is an isomorphism for all objects $C \in \text{ob}(\mathcal{C})$.

Example 3.24. If one restricts themselves to the category of finite vector spaces over k, then $\alpha : \text{id} \implies (-)^{**}$ is a natural isomorphism. This reflects the idea that there are canonical isomorphisms $V \cong V^{**}$ for all V (they are consistent with respect to morphisms $V \to W$ and the respective $V^{**} \to W^{**}$). Meanwhile, although $V \cong V^*$ as well, this requires a non-canonical choice of basis in each case making it impossible for the required diagrams to commute everywhere.

Remark. There is a generalization of categories known as **2-categories**, where in addition to objects and morphisms between objects, there is a notion of **2-morphisms** between morphisms. For example, **Cat** is a 2-category whose objects are small categories, whose morphisms are functors, and whose 2-morphisms are natural transformations.

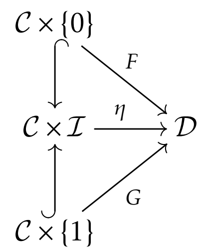
Exercise 3.25. *Let $\mathcal{I}$ be the **interval category***

$$\mathcal{I} = \{0 \to 1\}$$

*(it has two objects and one non-trivial morphism). Prove that a **natural transformation** is precisely a functor*

$$\eta : \mathcal{C} \times \mathcal{I} \to \mathcal{D}$$

such that the diagram of functors



commutes (this diagram lives in **Cat***). Note that* $\hom_{\mathcal{C}\times\mathcal{I}}((x,s),(y,t))$ *is given by* $\hom_{\mathcal{C}}(x,y) \times \hom_{\mathcal{I}}(s,t)$*.*

Remark. This exercise is more compelling when done after learning about homotopies. It asserts that natural transformations are their categorical analogue.

Definition 3.26. Categories $\mathcal{C}, \mathcal{D}$ are **equivalent categories** if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G \implies 1_D$ and $G \circ F \implies 1_C$ are natural isomorphisms of functors.

Remark. Continuing the analogy from the exercise, an equivalence of categories is the categorical analogue of being homotopy equivalent. This notion is more useful than a potential notion of isomorphism (e.g., where $F \circ G = 1_D$ exactly). Most importantly, an equivalence still preserves "categorical concepts" such as monomorphisms, limits, etc. (see `https://en.wikipedia.org/wiki/Equivalence_of_categories#Properties`).

Example 3.27. Let $\mathcal{C}$ be a category whose objects are non-negative integers and whose homomorphisms are

$$\hom_{\mathcal{C}}(m,n) = \{n \times m \text{ matrices in } \mathbb{R}\}.$$

Then $\mathcal{C}$ turns out to be equivalent to the category $\mathbf{Vect}_{\mathbb{R}}^{\text{finite}}$. Take F as a functor from m to the "standard" $\mathbb{R}^m$ and G as a functor from an n-dimensional $\mathbb{R}$-vector space to n. Matrices are taken to the corresponding linear maps, and vice versa.

Remark. This example exhibits that an equivalence of categories does not even ensure a bijection of objects. However, categorical concepts are preserved, which is what matters (e.g., think about the relative importance of isomorphism over equality).

Example 3.28. In algebraic geometry, the category of affine schemes is equivalent to the opposite category of commutative rings. Take F as a functor from an affine scheme to its ring of global sections, and take G as a functor from a commutative ring to its spectrum.

3.4 [DRAFT] Universal properties

Remark. This section is based on [1.5], Wikipedia, and the previously mentioned reference. We reframe via the notions of initial and final topologies as discussed by Wikipedia (not covered in class).

A **universal property** can be thought of as a description (with respect to other objects/morphisms) that uniquely defines an object in a category up to isomorphism, whereby the object can be thought of as the most "efficient solution" to a diagram (i.e., **universal with respect to the diagram**) such that all other solutions factor through it. This is terribly vague; a technical definition can be found at https://en.wikipedia.org/wiki/Universal_property. It is best to proceed by providing examples.

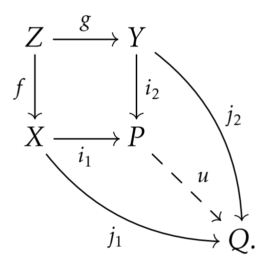
Example 3.29. **(TODO: Universal property of the free group)** https://en.wikipedia.org/wiki/Free_group

Example 3.30. We saw that the tensor product $M \otimes_R N$ of modules was defined by the universal property of being factored through by any R-bilinear maps from $M \times N$.

Definition 3.31. The **pushout** or **fibered sum** $X \cup_Z Y$ of morphisms $f : Z \to X$ and $g : Z \to Y$ is given by an object P and morphisms $i_1 : X \to P$, $i_2 : Y \to P$ such that

$$\begin{CD} Z @>g>> Y \\ @VfVV @VVi_2V \\ X @>>i_1> P \end{CD}$$

commutes, and that (P, i_1, i_2) is universal for this diagram; i.e., for any other solution (Q, j_1, j_2) making such a diagram commute, there exists a unique u that makes the following diagram commute:



Example 3.32. **(TODO: pushout of sets)**

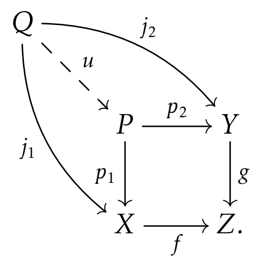
Example 3.33. **(TODO: pushout of groups is amalgamated product)**

Example 3.34. **(TODO: pushout of pointed spaces is wedge product)**

Definition 3.35. The **pullback** or **fiber product** $X \times_Z Y$ of morphisms $f : X \to Z$ and $g : Y \to Z$ is given by an object P and morphisms $p_1 : P \to X$, $p_2 : P \to Y$ such that

$$\begin{CD} P @>p_2>> Y \\ @Vp_1VV @VVgV \\ X @>>f> Z \end{CD}$$

commutes, and that (P, p_1, p_2) is universal for this diagram; i.e., for any other solution (Q, j_1, j_2) making such a diagram commute, there exists a unique u that makes the following diagram commute:



Example 3.36. **(TODO: pullback of sets)**

Example 3.37. **(TODO: pullback of rings)**

Definition 3.38. Let $f_i : X \to Y_i$ be a family of functions. The **initial topology** on X is the coarsest topology such that all f_i are continuous. It is characterized by the following property: a function $g : Z \to X$ is continuous if and only if $f_i \circ g : Z \to Y_i$ are all continuous.

(TODO: diagram)

Example 3.39. The **subspace topology** is the initial topology with respect to an inclusion map $A \to X$.

(TODO: diagram)

Example 3.40. The **product topology** is the initial topology with respect to the projection maps (the underlying space is the set product).

(TODO: diagram)

Definition 3.41. Let $f_i : X_i \to Y$ be a family of functions. The **final topology** on Y is the finest topology such that all f_i are continuous. It is characterized by the following property: a function $g : Y \to Z$ is continuous if and only if $g \circ f_i : X_i \to Z$ are all continuous.

(TODO: diagram)

Example 3.42. The **quotient topology** is the final topology with respect to a surjective map $X \to Y$.

(TODO: diagram)

Example 3.43. The **disjoint union topology** is the final topology with respect to the inclusion maps (the underlying space is the disjoint set union).

(TODO: diagram)

3.5 Adjoint functors

Definition 3.44. We say that $S : \mathcal{D} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ are an **adjoint pair of functors** (where S is **left adjoint** to T, or T is **right adjoint** to S) if for all $X \in \mathrm{ob}(\mathcal{D})$ and $Y \in \mathrm{ob}(\mathcal{C})$, we have bijections $\eta_{X,Y}$ via which

$$\hom_{\mathcal{C}}(SX, Y) \cong \hom_{\mathcal{D}}(X, TY)$$

such that the implicit transformations are natural in X and Y. Explicitly, for all X and Y, the pair of (covariant and contravariant, respectively) functors

$$\hom_{\mathcal{C}}(SX, -) \cong \hom_{\mathcal{D}}(X, T-)$$
$$\hom_{\mathcal{C}}(S-, Y) \cong \hom_{\mathcal{D}}(-, TY)$$

are naturally isomorphic.

Example 3.45. Consider the functors $- \otimes_R N : \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$ and $\mathrm{Hom}(N, -) : \mathbf{R}\text{-}\mathbf{Mod} \to \mathbf{R}\text{-}\mathbf{Mod}$. The tensor product is left adjoint to the Hom map; note that

$$\hom_{\mathbf{R}\text{-}\mathbf{mod}}(SX, Y) = \hom_{\mathbf{R}\text{-}\mathbf{mod}}(X \otimes_R N, Y) \cong \hom_{\mathbf{R}\text{-}\mathbf{mod}}(X, \mathrm{Hom}(N, Y)) = \hom_{\mathbf{R}\text{-}\mathbf{mod}}(X, TY)$$

via the tensor-hom adjunction we saw earlier.

Remark. This is a consequence of $\mathbf{R}$-**mod** equipped with

$$\otimes_R : \mathbf{R}\text{-}\mathbf{mod} \times \mathbf{R}\text{-}\mathbf{mod} \to \mathbf{R}\text{-}\mathbf{mod}$$

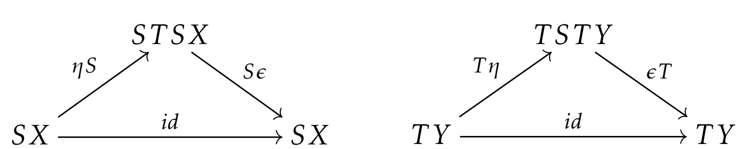
being a **closed (symmetric) monoidal category**. Such categories are characterized by the property that the associated "bifunctor" (in this case $\otimes_R$), when restricted to be a functor $- \otimes_R N$ for some fixed N, has a right adjoint functor denoted $[N, -]$. We will see other adjunctions of this type when discussing mapping spaces.

Example 3.46. Let $S : \mathbf{Set} \to \mathbf{Grp}$ be the functor that takes a set to its free group, and let $T : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor (taking the group to its underlying set). Then S is left adjoint to T. To see this, the desired assertion is that for any $X \in \mathbf{Set}$ and $Y \in \mathbf{Grp}$,

$$\hom_{\mathbf{Grp}}(SX, Y) = \hom_{\mathbf{Grp}}(F_X, Y) \cong \hom_{\mathbf{Set}}(X, |Y|) = \hom_{\mathbf{Set}}(X, TY),$$

and the isomorphism follows directly from the universal property of the free group.

Exercise 3.47. *Let $\epsilon : TS \implies 1_{\mathcal{D}}$ and $\eta : 1_{\mathcal{C}} \implies ST$ be natural transformations. Show that the diagrams*



commute for all X, Y if and only if T, S are adjoint.

4 The fundamental group

Remark. Corresponding readings are [2.1-2.2], though these are not comprehensive. The presentation here is primarily from class.

4.1 Connectedness and paths

Definition 4.1. X is **connected** if X cannot be written as the disjoint union of two non-empty open sets.

Definition 4.2. X is **path-connected** if for all $x, y \in X$, there exists a continuous function $p : [0,1] \to X$ with $p(0) = x$ and $p(1) = y$ (a **path** between x and y).

Proposition 4.3. *Path-connectedness implies connectedness.*

Proof. Suppose X is path-connected but not connected, with $X = U \sqcup V$ disjoint non-empty open sets. Let $x \in U$ and $y \in V$, and let $p : [0,1] \to X$ be the path connecting them. Then

$$[0,1] = p^{-1}(X) = p^{-1}(U \sqcup V) = p^{-1}(U) \sqcup p^{-1}(V),$$

a disjoint non-empty union of open sets. This contradicts the fact that $[0,1]$ is connected. □

Proposition 4.4. *The existence of a path is an equivalence relation on points in X.*

Proof. Reflexivity is given by taking $p : [0,1] \to X$ to be the constant map to x. Transitivity is given by gluing paths together (re-parameterizing so that the domain is still $[0,1]$). Symmetry is given by taking $q(x) := p(1-x)$. □

Proposition 4.5. *A union of (path-)connected spaces with a point in common are (path-)connected. The image of a (path-)connected space under a continuous map is (path-)connected.*

Proof. For connectedness:

a) Let X, Y be connected and consider $X \cup Y$ with a common element x (implicit here is that the subspace topologies on X and Y give their original topology). The key is to argue that for one of the connected $Z = X, Y$, the disjoint union $(A \cap Z) \sqcup (B \cap Z)$ is a non-empty (and necessarily disjoint, open by relativity) partition of Z, a contradiction. There exists a common element, it is WLOG in A (and thus it is in every $A \cap Z$); then, choose the other element in $B \cap Z$ (which is non-zero for some Z).

b) The disjoint non-empty open partition of a space would also partition its pre-image.

For path-connectedness:

a) Construct paths in the new space by passing through the shared point as needed.

b) The image of a path is a path.

These observations prove the corresponding statements. □

Example 4.6 (Topologist's sine curve)**.** The converse is not true, even in a metric space. Furthermore, path-connectedness is not preserved by closure (though connectedness is). Consider the graph $\Gamma = \{(x, \sin(1/x)) \mid x \in (0,1]\} \subseteq \mathbb{R}^2$. Its closure is

$$\overline{\Gamma} = \Gamma \cup (\{0\} \times [-1,1]).$$

is closed and bounded and also compact. Pre-closure, we had the graph of a continuous function, which is path-connected (e.g., parameterize with $(0,1]$). However, its closure remains connected, but it is not locally connected nor path-connected. Problems occur around $(0,0)$. See http://math.stanford.edu/~conrad/diffgeomPage/handouts/sinecurve.pdf for details.

Definition 4.7. The equivalence relation of path-connectedness induces a partition of X into **path-connected components**. We denote the set of path components as $\pi_0(X)$.

4.2 Homotopy and homotopy equivalence

Definition 4.8. Two continuous maps $f, g : X \to Y$ are **homotopic**, written $f \simeq g$, if there exists a continuous $h : X \times I \to Y$ (called a **homotopy**) such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$.

Exercise 4.9. *Homotopy is a equivalence relation on continuous maps $X \to Y$.*

Example 4.10. Imagine two maps f, g from $X = [0,1]$ to two curves in a space Y. Then we have a homotopy which continuously transforms the image from the first curve to the second. Note that this can fail when there is a "hole".

Definition 4.11. A function f is **null-homotopic** if it is homotopic to a constant function.

Definition 4.12. Two spaces X, Y are **homotopy equivalent** (written $X \simeq Y$) if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.

Remark. Compare this definition with X, Y being homeomorphic, where we need $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Exercise 4.13. *Homotopy equivalence is a equivalence relation on topological spaces.*

Example 4.14. Let $X = \mathbb{R}^2$, $Y = \{0\}$. We want to show the two are homotopy equivalent. Let

$$\begin{aligned} f: \quad & X \to Y, \quad f(x,y) = 0 \\ g: \quad & Y \to X, \quad g(0) = (0,0). \end{aligned}$$

Then $f \circ g = id_Y$, so $f \circ g \simeq id_Y$. Meanwhile, $(g \circ f)(\vec{v}) = (0,0)$. We need $g \circ f \simeq id_X$. Consider the following null-homotopy:

$$H : X \times [0,1] \to X, \quad H(\vec{v}, t) = t\vec{v}.$$

Then $H(\vec{v}, 0) = 0\vec{v} = (0,0)$ and $H(\vec{v}, 1) = id_X$, exhibiting that $g \circ f \simeq id_X$. We can visualize this as the continuous contraction of the space to the origin.

Intuitively, homotopy equivalence indicates that two spaces can be transformed into each other by bending, shrinking, and expanding. Homeomorphism implies homotopy equivalence, but not vice versa (the disk and the point are homotopy equivalent by shrinking along radial lines, but not homeomorphic since there is certainly no bijection).

Definition 4.15. X is **contractible** if it is homotopy equivalent to the space of one element. One can show this is equivalent to $\mathrm{id}_X : X \to X$ being null-homotopic.

Example 4.16. Let $X = \mathbb{R}^2 \backslash \{(0,0)\}$ and $Y = S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Note that $Y \subseteq X$. We want to show these are homotopy equivalent. Let

$$\begin{aligned} g : Y \to X, \quad & g(\vec{v}) = \vec{v} \\ f : X \to Y, \quad & f(\vec{v}) = \frac{\vec{v}}{\|\vec{v}\|}. \end{aligned}$$

Then $f \circ g = id_Y$ and $(g \circ f)(\vec{v}) = \frac{\vec{v}}{\|\vec{v}\|}$. We show the latter is homotopic to id_X via:

$$H : X \times [0,1] \to X, \quad H(\vec{v}, t) = t\vec{v} + (1-t)\frac{\vec{v}}{\|\vec{v}\|}.$$

Then $H(\vec{v}, 0) = \frac{\vec{v}}{\|\vec{v}\|}$ and $H(\vec{v}, 1) = \vec{v}$.

Remark. Note that homotopy equivalence is weaker than homeomorphism. In this previous example, we saw the homotopy equivalence of a non-compact space and a compact space. What let this argument pass through is that in some sense, homotopy equivalence preserves the number of holes but not other topological properties.

4.3 The fundamental group

Definition 4.17. Let $x \in X$. A **loop at** b is a path $p : [0,1] \to X$ with $p(0) = p(1) = b$.

Remark. A **Moore loop** is a variant definition of a loop which attaches a domain $\mathbb{R}/n\mathbb{Z}$ to a loop. The necessary conclusions (e.g., the fundamental group) turn out to be equivalent.

Remark. We adopt the following convention: A **homotopy of loops** p, q at b has to fix the endpoints at b. Thus,

$$H : [0,1] \times [0,1] \to X$$

(the first $[0,1]$ represents the path, the second $[0,1]$ represents time), so that

$$H(x,0) = p(x), \quad H(x,1) = q(x), \quad H(0,t) = H(1,t) = b.$$

Definition 4.18. The **fundamental group** $\pi_1(X,\star)$ is the group produced by taking the set of loops at $\star$, modulo the equivalence relation of loop homotopy.

- The group operation is induced via **loop concatenation**:

$$(p * q)(t) = \begin{cases} p(2t) \text{ if } 0 \le t \le \frac{1}{2} \\ q(2t-1) \text{ if } \frac{1}{2} \le t \le 1. \end{cases}$$

- This only gives e.g., associativity if we work *modulo homotopy*. One can check that

$$(p * q) * r \simeq p * (q * r)$$

  explicitly, as desired.

- Our identity element is (the equivalence class) of the constant loop at $\star$, which we call $c_\star$. One can construct a homotopy $p * c_\star \simeq p$, for example.

- The inverse element p^{-1} of a loop p can be defined as $p^{-1}(t) := p(1-t)$. One can construct a homotopy whereby one traverses less and less of the path until it becomes the constant map [HW1, Q1].

If $[p]$ denotes the equivalence class of p, then it turns out

$$[p] \cdot [q] := [p * q]$$

is a well-defined group operation. That is, loop concatenation $*$ gives a group operation *up to homotopy*.

Definition 4.19. Let $f : X \to Y$ where p is a loop at $\star \in X$. Then $f \circ p$ is a loop at $f(\star) \in Y$. In this way, f induces the **induced homomorphism** written

$$f_* : \quad \pi_1(X,\star) \to \pi_1(Y, f(\star)), \quad [p] \mapsto [f \circ p].$$

Remark. See [HW1, Q6] where we check that the fundamental group and the induced homomorphism have the desired properties:

1) is the assertion that $[p][q] = [p * q]$ is well-defined.

2) is the assertion that $f_*([p]) = [f \circ p]$ is well defined.

3) is the assertion that f_* is a homomorphism (i.e., $f_*([p][q]) = f_*([p])f_*[q])$).

4) is the assertion that $f \simeq g$ implies $f_* = g_*$.

Corollary 4.20. *The following hold from the property $f \simeq g$ implies $f_* = g_*$:*

- *If $f : X \to Y$ is a homotopy equivalence, then f_* is an isomorphism (i.e., bijective group homomorphism) [HW1, Q7]*
- *Homotopy-equivalent spaces have the same fundamental group.*
- *A contractible space has trivial fundamental group.*

Example 4.21 (Fundamental group of S^1)**.** We claim that

$$\pi_1(S^1, (1,0)) \cong \mathbb{Z}.$$

We sketch the proof visually: consider the real line coiled over S^1, such that $\mathbb{Z}$ is the preimage of $(1,0)$. Following the loop along the real line, we end at one of these integers. If we class the loops based on which of $\mathbb{Z}$ our loop ends up at, this turns out to construct an isomorphism between the homotopy classes of loops and $\mathbb{Z}$. See `https://www.math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Dooley.pdf`, which formalizes this via path lifting and covering spaces.

Example 4.22. The fundamental group is a functor

$$\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}.$$

The morphisms are mapped via

$$f : (X, x) \to (Y, y) \quad \mapsto \quad f_* : \pi_1(X, x) \to \pi_1(Y, y)$$

and composition is preserved since $(g \circ f)_* = g_* \circ f_*$.

Definition 4.23. The **fundamental groupoid** ΠX of a space X is the category whose objects are points in X, and whose morphisms are

$$\Pi X(x, y) = \{\text{paths from } x \text{ to } y\}/\{\text{homotopy}\}.$$

This is a groupoid since the existence of return paths exhibits isomorphism. Note that $\Pi X(x, x)$ forms a group isomorphic to $\pi_1(X, x)$.

4.4 Applications

Remark. The following are Theorems 1.8 and 1.9 in Hatcher.

Theorem 4.24 (Fundamental theorem of algebra). *A polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ has a root in $\mathbb{C}$ if $n \geq 1$.*

Proof.
- Suppose

  $$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

  has no roots. Define the family of functions

  $$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{\left|p(re^{2\pi is})/p(r)\right|}.$$

  where $r \geq 0$.

- Then $|f_r(s)| = 1$ (the function is well-defined everywhere, and no denominator is zero by hypothesis) and $f_r(0) = f_r(1) = 1$, making f_r a loop in $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ based at $1 \in \mathbb{C}$.

- We claim that $[f_0] = [f_r] = 0 \in \pi_1(S^1, 1)$: First note that $f_0(s) = 1$, i.e., the constant function (equivalently, view f_0 as the constant loop at 1). Then $[f_0] = [f_r]$ with the homotopy exhibited by f_r varying from 0 to r (continuous since composition of continuous functions like multiplication, norm, non-zero division, etc.). That is, every f_r is homotopic to the trivial loop.

- Now, define

  $$p_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)$$

  and note that $p_0(z) = z^n$ and $p_1(z) = p(z)$, exhibiting a homotopy $z^n \simeq p(z)$. By the reverse triangle inequality:

  $$|p_t(z)| \geq |z|^n - \left|t(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)\right|$$

- For $|z| = r$ sufficiently large ($r \geq \max\{|a_{n-1}| + \cdots + |a_0|, 1\}$), we have the inequality

  $$|z|^n = r \cdot r^{n-1} \geq (|a_{n-1}| + \cdots + |a_0|)\,|z|^{n-1} \geq \left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right|$$

  (the last inequality holds since $|z| = r \geq 1$). Then for $t \in [0,1]$, we have $|p_t(z)| > 0$ when $|z| = r$.

- Since $p_t(z) \neq 0$ if $|z| = r$ for all $t \in [0,1]$, we fix this r which allows us take the following homotopy (no divisions by zero anywhere)

$$\begin{aligned}
&\implies \frac{p_0(re^{2\pi is})/p_0(r)}{\left|p_0(re^{2\pi is})/p_0(r)\right|} \simeq \frac{p_1(re^{2\pi is})/p_1(r)}{\left|p_1(re^{2\pi is})/p_1(r)\right|} = \frac{p(re^{2\pi is})/p(r)}{\left|p(re^{2\pi is})/p(r)\right|} \\
&\implies e^{2\pi ins} \simeq f_r(s).
\end{aligned}$$

- The left-hand side as a function of s corresponds to the class $n \in \pi_1(S^1, 1)$. But we saw that $[f_r] = [f_0] =\in \pi_1(S^1, 1)$. Our polynomial is not constant, so $n \neq 0$, a contradiction.

□

Theorem 4.25 (Brouwer's fixed point theorem)**.** *Let*

$$\begin{aligned}
D^n &= \{\vec{v} \in \mathbb{R}^n \mid \left\|\vec{v}\right\| \leq 1\} \\
S^{n-1} &= \{\vec{v} \in \mathbb{R}^n \mid \left\|\vec{v}\right\| = 1\}
\end{aligned}$$

*and take $f : D^n \to D^n$ continuous. Then there exists $v \in D^n$ such that $f(v) = v$, i.e., a **fixed point**.*

Proof. We complete the $n = 2$ case. Assume f does not fix a point.

- Define $g : D^2 \to S^1$ such that the g goes to the intersection of the ray $\overrightarrow{f(\vec{v})\vec{v}}$ with S^1 (this is well-defined since $f(\vec{v}) \neq \vec{v}$ everywhere). Observe that g is continuous (informally, f is continuous and so the ray produced would vary continuously with $\vec{v}$).

- If $\vec{v} \in S^1$, we have $g(\vec{v}) = \vec{v}$. If ι denotes inclusion, then we have

$$S^1 \xrightarrow{\iota} D^2 \xrightarrow{g} S^1$$

  where $g \circ i = \mathrm{id}_{S^1}$.

- Passing to fundamental groups, we get

$$\pi_1(S^1) \xrightarrow{\iota_*} \pi_1(D^2) \xrightarrow{g_*} \pi_1(S^1).$$

  Since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(D^2) \cong 0$ (by contractibility, see [HW1, Q4]), then $(g \circ i)_* = (\mathrm{id}_{S^1})_* = id_{\mathbb{Z}}$, but that is impossible since it must factor through the trivial group $\pi_1(D^2)$. This is a contradiction.

For the cases $n > 2$, one uses π_n or the homology groups (see [Hatcher, Corollary 2.11]).

□

Exercise 4.26. *Prove the case $n = 1$ using the intermediate value theorem.*

Proof. We want $f : D^1 = [0,1] \to D^1 = [0,1]$ continuous. Suppose we have no fixed point; then $g(t) := f(t) - t \neq 0$ on our domain. $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$. By the intermediate value theorem, there is some value $t \in [0,1]$ for which $g(t) = 0$, a contradiction. (For a visual picture, draw the graphs of $\mathrm{id}_{[0,1]}$ and an arbitrary f, and note that they always intersect.) □

4.5 [TODO] Van Kampen's theorem

Remark. **(TODO: A brief shoutout to Van Kampen's theorem for fundamental groups, plus an example of a pushout: `http://www.math.toronto.edu/mat1300/vankampen.pdf`, or Hatcher)**

5 Higher homotopy groups

5.1 Mapping spaces

Remark. The unbased mapping space properties described below are covered in [2.2]. The conditions from which we get the adjunction relation are given by Hatcher.

Definition 5.1. The **mapping space** of X, Y is the topological space

$$\mathrm{Map}(X,Y) = Y^X = \{\text{continuous functions } X \to Y\}$$

with the **compact-open topology**: for every pair of compact sets $C \subseteq X$ and open sets $U \subseteq Y$, let $W(C,U)$ be the set of functions f with $f(C) \subseteq U$, then take the topology for which the set of all $W(C,U)$ is a subbasis.

Remark. If X is compact and Y has a metric, then Y^X has a metric

$$d(f,g) = \sup_{x \in X} d(f(x), g(x))$$

which induces the same topology as the compact-open topology.

Example 5.2. Let X, Y, Z be spaces, with Y locally compact. Then

$$\mathrm{Map}(X \times Y, Z) = Z^{X \times Y} \cong (Z^Y)^X = \mathrm{Map}(X, \mathrm{Map}(Y,Z)),$$

a homeomorphism. The elements of these spaces look like

$$f : X \times Y \to Z, \quad g : X \to Z^Y.$$

The idea is that for every $x \in X$ fed into f, one induces a function $Y \to Z$. Thus g is a function from this x to the functions $Y \to Z$. The bijective correspondence is observable by noting that each can be defined in terms of the other:

$$(x,y) \mapsto (g(x))(y) \quad \leftrightarrow \quad x \mapsto (y \mapsto f(x,y)).$$

This holds unconditionally if we were considering arbitrary functions. However, we now work on the mapping *spaces* with compact-open topologies. It turns out that Y's local compactness is strong enough to ensure that f is continuous if and only if g is continuous, and that the compact-open topology allows for a bijection that is a homeomorphism.

Remark. This parallels the tensor-hom adjunction, in the sense that

$$- \times Y : \mathbf{Top} \to \mathbf{Top}, \quad \mathrm{Map}(Y,-) : \mathbf{Top} \to \mathbf{Top}$$

are an adjoint pair of functors for Y locally compact. The parallel is more appropriate if we did not have to make such a specific restriction; instead, if we restricted X,Y,Z to be objects in the category **CGWH** of compactly-generated weak Hausdorff spaces (see [*Concise*, §5] and http://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf), then the bijection holds without restriction (as in the case of tensor-hom).

Definition 5.3. The **based mapping space** of $(X,\star_x),(Y,\star_y)$ is

$$\mathrm{Map}_\star((X,\star_x),(Y,\star_y)) = \{\text{continuous functions } X \to Y \text{ mapping } \star_x \text{ to } \star_y\}.$$

equipped with the compact-open topology.

Example 5.4. Analogous to (unbased) mapping spaces, under the compact-open topology and the constraint that Y is locally compact, one gets the homeomorphism

$$\mathrm{Map}_\star((X \wedge Y,\star),(Z,\star_z)) \cong \mathrm{Map}_\star((X,\star_x),\mathrm{Map}_\star((Y,\star_y),(Z,\star_z)))$$

where $\wedge$ denotes the **smash product** defined below.

Remark. As with the unbased case, we can abandon the constraint of Y being locally compact if X,Y,Z live in a nice category like $\mathbf{CGWH}_*$. More formally, $\wedge$ makes $\mathbf{CGWH}_*$ a closed symmetric monoidal category (see the section on adjoint functors).

5.2 Operations on pointed spaces

Remark. The following is a synthesis of part of Chapter 0 in Hatcher, along with the readings and the class' content.

Notation 5.5. When the existence of a basepoint is obvious and necessary, we might simply write X to denote the pointed space $(X,\star_x)$.

Definition 5.6. The **wedge product**, written $X \vee Y$, of two pointed spaces $(X, \star_x)$ and $(Y, \star_y)$ is the quotient of the disjoint union $X \sqcup Y$ given by identifying $\star_x \sim \star_y$ (this is necessary since working with basepoints isn't canonical for disjoint unions).

Example 5.7. The wedge product $X \vee Y$ is the pushout in the following diagram:

$$\begin{array}{ccc} \{\star\} & \xrightarrow{\star_y} & Y \\ {\scriptstyle \star_x}\downarrow & & \downarrow{\scriptstyle i_2} \\ X & \xrightarrow[i_1]{} & X \vee Y. \end{array}$$

Definition 5.8. The **smash product**, written $X \wedge Y$, of two pointed spaces $(X, \star_x)$ and $(Y, \star_y)$ is the quotient of the product space $X \times Y$ given by identifying $(x, \star_y) \sim (\star_x, y)$ for all x, y. These are copies of X and Y that only intersect at $(\star_x, \star_y)$. More precisely,

$$X \wedge Y := (X \times Y)/(X \vee Y).$$

Example 5.9. $S^1 \times S^1$ is the 2-torus, but $S^1 \wedge S^1 = S^2$. In general, $S^m \wedge S^n \cong S^{m+n}$. A nice way to visualize this is to consider the torus as a square with its opposite sides identified. These sides form two circles intersecting at a point, a copy of $S^1 \vee S^1$. When they are identified to be one point (that is, $(S^1 \times S^1)/(S^1 \vee S^1)$, the square becomes the sphere S^2.

Example 5.10. We saw earlier that the smash product gave a homeomorphism of based mapping spaces, analogous to the tensor product and the tensor-hom adjunction. Similarly, the smash product satisfies the universal property that every map from $(X, \star_x) \times (Y, \star_y)$ that preserve basepoints separately in each variable factors through the smash product:

$$\begin{array}{ccc} (X, \star_x) \times (Y, \star_y) & \xrightarrow{\wedge} & (X \wedge Y, \star) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \tilde{f}} \\ & & (Z, \star_z) \end{array}$$

(see http://mathoverflow.net/a/105833).

Definition 5.11. Given a *non-pointed* space X, the **suspension** SX is the quotient of $X \times I$ where $X \times \{0\}$ is collapsed to one point and $X \times \{1\}$ is collapsed to another.

Example 5.12. The motivating example is $X = S^1$. Then SX can be viewed as a double-pointed cone with base X, from which we see that $S(S^1) = S^2$ with the two "suspension points" as $(0, 0, \pm 1)$. This argument generalizes in the obvious way for $X = S^n$.

Remark. For continuous $f : X \to Y$, we can define $Sf : SX \to SY$ to be $Sf([x,t]) = [f(x),t]$ (the brackets indicate that we are working modulo equivalence). Then we have the **suspension functor**

$$S : \mathbf{Top} \to \mathbf{Top}$$

which roughly increases the dimension of a space by one.

Definition 5.13. Given a pointed space $(X, \star_x)$, the **reduced suspension** is given by

$$\Sigma X := (X \times I)/(X \times \{0\} \cup X \times \{1\} \cup \{\star_x\} \times I),$$

or equivalently, SX with the line joining the two suspension points also identified. The identification is done to ensure there is a canonical basepoint, namely the equivalence class of all the identified points.

Remark. Likewise we have the **reduced suspension functor**

$$\Sigma : \mathbf{Top}_* \to \mathbf{Top}_*$$

which roughly increases the dimension of a pointed space by one.

Proposition 5.14. $\Sigma X \cong X \wedge S^1$

Proof. Briefly: taking $X \times I$ and then identifying $(x,0) \sim (x,1)$ gives you $X \times S^1$. The further identification of $X \times \{0\} = X \times \{1\}$ is like collapsing along X, and the further identification of $\{\star_X\} \times I$ is like collapsing along S^1. This gives you

$$\begin{aligned}
&((X \times I)/(x,0) \sim (x,1))/(X \times \{0\} \cup \{\star_x\} \times I) \\
&\cong (X \times S^1)/(X \times \{0\} \cup \{\star_x\} \times I) \\
&\cong (X \times S^1)/(X \vee S^1),
\end{aligned}$$

which is just $X \wedge S^1$. $\square$

Definition 5.15. Given a pointed space $(X, \star_x)$, its **loop space** is given by

$$\Omega X := \mathrm{Map}_\star(S^1, X),$$

that is, the based mapping space of maps from S^1 to X. This gives the **loop functor** on pointed spaces, whereby for $f : X \to Y$ we take $\Omega f : \mathrm{Map}_\star(S^1, X) \to \mathrm{Map}_\star(S^1, Y)$ to be (post-)composition with f.

Proposition 5.16.

$$\begin{aligned}
Map_\star(\Sigma X, Y) &\cong Map_\star(X, \Omega Y) \\
Map_\star(X \wedge I_\star, Y) &\cong Map_\star(I_\star, Map_\star(X, Y)).
\end{aligned}$$

Proof. Both follow from the adjunction property for based mapping spaces that we saw in the previous section, along with the definitions of Σ, Ω and the symmetric nature of the smash product. $\square$

Example 5.17. We conclude that $\Sigma : \mathbf{Top}_* \to \mathbf{Top}_*$ is a left adjoint to the functor $\Omega : \mathbf{Top}_* \to \mathbf{Top}_*$.

5.3 Higher homotopy groups

Our observations regarding based mapping spaces and associated operations on pointed spaces (the wedge and smash product, along with loop and reduced suspension spaces) let us define the homotopy groups in full generality.

Definition 5.18. The **based mapping space up to homotopy** for based maps $X \to Y$ is denoted $[X, Y]$ and given by $\mathrm{Map}_\star(X, Y)$ modulo homotopy equivalence.

Definition 5.19. The n**-th homotopy group** is defined by

$$\pi_n(X) := [S^n, X].$$

This generalizes the definition of $\pi_1(X)$ as based loops up to homotopy.

Proposition 5.20. *Every higher homotopy group is the fundamental group of some space; namely*

$$\pi_n(X) = \pi_1(\Omega^{n-1} X).$$

Proof. Note that

$$\mathrm{Map}_\star(S^n, X) \cong \mathrm{Map}_\star(S^1 \wedge S^{n-1}, X) \cong \mathrm{Map}_\star(S^1, \mathrm{Map}_\star(S^{n-1}, X)),$$

where we use our based mapping homeomorphism and the smash product of spheres. Taking our mapping spaces modulo homotopy gives

$$\pi_n(X) \cong \pi_1(\Omega^{n-1} X).$$

in a canonical way (the homeomorphism is induced from the based mapping homeomorphism, which expresses a pair of adjoint functors). $\square$

Remark. It follows that the higher homotopy groups are also functors

$$\pi_n : \mathbf{Top}_* \to \mathbf{Grp},$$

being compositions of the functors Ω and π_1.

Proposition 5.21. π_n *is an abelian group for* $n \geq 2$.

Proof.
- In a nice category of spaces (e.g., **CGWH**), one can show that ΩX is something known as an H**-space**, also known as a **topological unital magma**. The result that the fundamental group of a topological group is abelian generalizes to H-spaces, and since $\pi_n(X) = \pi_1(\Omega^{n-1}X)$ we have the result.
- Without assuming a nice category, note that it suffices to consider $n = 2$ (since $\pi_n(X) = \pi_2(\Omega^{n-2}(X))$). It suffices to consider $n = 2$ (since $\pi_n(X) = \pi_2(\Omega^{n-2}(X))$). Homotopy classes of maps $S^2 \to X$ are equivalent to classes of maps $I^2 \to X$ that take the boundary to the basepoint. The essential idea is that a concatenation $f * g : I^2 \to X$ can be deformed via the extra dimension into $g * f : I^2 \to X$ (see http://math.stackexchange.com/a/161519).

□

Example 5.22. For the n-sphere S^n, we have

$$\pi_q(S^n) = 0 \text{ for } q < n, \quad \pi_n(S^n) = \mathbb{Z}.$$

The last statement is not easy to prove, and is most immediately viewed as a consequence of a result known as the Hurewicz theorem. There are also:

$$\begin{aligned} \pi_{4n-1}(S^{2n}) &= \mathbb{Z} \oplus \text{finite} \\ \pi_q(S^n) &= \text{ is finite otherwise and non-zero for infinitely many } q > n,\ n \geq 2. \end{aligned}$$

Definition 5.23. Let $f : X \to Y$. This induces $f_* : \pi_n(X) \to \pi_n(Y)$ (think $S^n \to X \to Y$ where the second map is f). We say that f is a **weak homotopy equivalence** if f_* is an isomorphism for all $n \geq 0$.

Proposition 5.24. *Homotopy equivalence implies weak homotopy equivalence.*

Proof. Suppose $X \simeq Y$ via maps $f : X \to Y$ and $g : Y \to X$. For $n = 0$, note that homotopy equivalence preserves the number of path-connected components. For $n \geq 1$, since $f \circ g \simeq \text{id}_X$, we have $g_* \circ f_* = (g \circ f)_* = \text{id}_{X*}$, we conclude that f_* is an isomorphism of groups. □

Remark. For "nice spaces" like CW-complexes, the converse is true. Furthermore, non-nice-spaces are weak homotopy equivalent to nice spaces.

Homotopy groups are more sophisticated invariants than the homology groups and thus better at distinguishing spaces. However, they are hard, if not impossible, to compute. Meanwhile, homology groups will turn out to be independent of the decomposition we choose for turning a space into a simplicial complex.

6 Simplicial complexes

Remark. This is a synthesis of Section 4 of May's *Finite Spaces* book, Munkres' *Elements of Algebraic Topology*, and things mentioned in class.

Ultimately we want to associate algebraic objects to arbitrary spaces. We start by considering spaces that are homeomorphic to a net of edges, triangles, tetrahedra, etc. (e.g., polyhedra). Many spaces (e.g., the torus) can be viewed in this way, leading to the notion of simplicial complexes.

6.1 Simplicial complexes

Definition 6.1. A set of points $\{v_0, \dots, v_n\}$ is **geometrically independent** if the vectors $\{v_i - v_0\}$ are linearly independent. One verifies that this is equivalent to the requirement that

$$\sum_{i=0}^{n} t_i = 0 \text{ and } \sum_{i=0}^{n} t_i v_i = 0 \implies t_0 = t_1 = \cdots = t_n = 0.$$

Definition 6.2. The n**-simplex** σ spanned by a geometrically independent set $\{v_0, \dots, v_n\}$ is the **convex hull** of the points, i.e.,

$$\sigma = \left\{ \sum_{i=0}^{n} t_i v_i \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}.$$

Simplices spanned by a subset of $\{v_0, \dots, v_n\}$ are **faces** of σ. The set $\{v_0, \dots, v_n\}$ is the **vertex set** of σ. The **dimension** of σ is n.

Definition 6.3. A **(geometric) simplicial complex** K is a collection of simplices in $\mathbb{R}^N$ where:

- Every face of a simplex in K is also a simplex in K
- The intersection of two simplices in K is also a simplex in K

The **vertices** $V(K)$ of a simplicial complex is the union of its simplices' vertex sets; i.e., it is the set of 1-simplices in K.

Example 6.4. The hollow tetrahedron with points at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ is a simplicial complex. The simplices are every proper, non-empty subset of these four points. The four points are geometrically independent (for example, take $v_0 = (0,0,0)$) and thus all the subsets define simplices. Closure under taking faces and intersections is immediate.

Definition 6.5. A **map of simplicial complexes** $f : K \to L$ is a map $V(K) \to V(L)$ where $f(\sigma)$ is a simplex in L for all simplices $\sigma \in K$.

Definition 6.6. The n**-skeleton** K^n of a simplicial complex is the simplicial complex determined by $K_0, \dots, K_n$, i.e., K's simplices of dimension up to n.

Definition 6.7. The **geometric realization** $|K|$ of a simplicial complex K is the topological space where:

- The underlying set is the union of the simplices of K
- The topology is that $V \subseteq |K|$ is closed exactly when $V \cap \sigma$ is closed in σ's subspace topology, for all $\sigma \in K$.

Remark. It turns out that if K is a finite collection of simplices, the subspace topology of $|K|$ as a subset of $\mathbb{R}^N$ is exactly the same as the topology defined above. See [Munkres, §2] to see that $|K|$'s topology is finer than the subspace topology in general.

Definition 6.8. The **category of simplicial complexes**, **SCxs** has geometric simplicial complexes as its objects and maps of simplicial complexes as its morphisms.

Definition 6.9. The **geometric realization functor**

$$|\cdots| : \mathbf{SCxs} \to \mathbf{Top}$$

takes simplicial complexes to their geometric realizations, and a map $g : K \to L$ of simplicial complexes to the continuous map

$$|g| : |K| \to |L|, \quad \sum t_i v_i \mapsto \sum t_i g(v_i).$$

6.2 Abstract simplicial complexes

Definition 6.10. An **abstract simplicial complex** X is comprised of the following data:

- A set X_0 whose elements we call **vertices**,
- Sets X_n of $(n+1)$-element subsets of X_0, whose elements we call n**-simplices**.

The sets of n-simplices are subject to the requirement that every $(k+1)$-subset of the vertices (a **face**) of an n-simplex (an element of X_n) is also a k-simplex (an element of X_k)

Proposition 6.11. *Every simplicial complex K determines an abstract simplicial complex aK.*

Proof. Take the vertices of K as X_0, and take each X_n to be the collection of $(n+1)$-element subsets of K that determine n-simplices. The requirement of subsets being abstract simplices is satisfied by the corresponding requirement on simplicial complexes that faces are also simplexes. □

Example 6.12. Consider an octahedron in $\mathbb{R}^3$ with only the top four faces filled in. Numbering the vertices from top to bottom and passing to its abstract simplicial complex, we have:

$$\begin{aligned}
X_0 &= \{x_1, \dots, x_6\} \\
X_1 &= \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_5\}, \\
&\quad \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_2\}, \\
&\quad \{x_2, x_6\}, \{x_3, x_6\}, \{x_4, x_6\}, \{x_5, x_6\}\} \\
X_2 &= \{\{x_1, x_2, x_3\}, \{x_1, x_3, x_4\}, \{x_1, x_4, x_5\}, \{x_1, x_5, x_2\}\}.
\end{aligned}$$

In this way we can see intuitive geometric relations encoded abstractly as subsets of the vertex set.

Remark. In the case of finitely many simplices, one can go in reverse. That is, one can pass from an abstract simplicial complex K to a simplicial complex gK by bijecting from X_0 to any geometrically independent subset of some $\mathbb{R}^N$ (this induces a bijection on each X_n). Then agK is isomorphic to K in the sense of abstract simplicial complexes (there is a bijection that preserves the relations between each's simplex sets X_i).

Definition 6.13. A **map of abstract simplicial complexes** $f : K \to L$ is a map $K_0 \to L_0$ where $f(\sigma)$ is a simplex in L for all simplices $\sigma \in K$ (i.e., σ is a subset of K_0 and thus $f(\sigma)$ is induced by the map on K_0; it should be an element of some L_i).

Definition 6.14. The **category of abstract simplicial complexes**, **AbsSCxs** has abstract simplicial complexes as its objects and maps of abstract simplicial complexes as its morphisms.

6.3 Ordered simplicial complexes

Definition 6.15. We have the following types of **binary relations** on a set S (i.e., subsets of $S \times S$):

- A **preorder** is a binary relation that is transitive and reflexive.
- A **partial order** is a preorder that is **antisymmetric** ($x \le y$ and $y \le x$ implies $x = y$).
- A **total order** is a partial order that is **total** (for all x, y, at least one of $x \le y$ or $y \le x$ must hold).

Definition 6.16. The **category of posets**, **Posets**, has sets equipped with partial orders (called **posets**) as its objects, and set maps satisfying

$$a \le b \implies f(a) \le f(b)$$

(called **order-preserving maps**) as its morphisms.

Definition 6.17. An **ordered simplicial complex** is an abstract simplicial complex equipped with a partial ordering on X_0 that restricts to a total order on each simplex. One can now write n-simplices $(x_0, \ldots, x_n)$ as having a total order $x_0 \leq \cdots \leq x_n$

Example 6.18. Consider the simplicial complex (of 0 and 1-simplices) given by the tree structure of PhD advisors and advisees. There is a natural partial ordering on the vertices as induced by the tree hierarchy. This ordering restricts to a total order on each of the 1-simplices (namely, advisor $\geq$ advisee).

Definition 6.19. The **category of ordered simplicial complexes**, **OrdSCxs** has ordered simplicial complexes as its objects and maps of abstract simplicial complexes that are order-preserving on the poset of vertices.

Example 6.20. There is a forgetful functor

$$\mathbf{OrdSCxs} \to \mathbf{AbsSCxs}$$

which "forgets" the partial ordering. For example, the two ordered simplicial complexes with simplices

$$\begin{aligned} X_0 &= \{a, b, c\}, \\ X_1 &= \{a, b\}, \{b, c\}, \{a, c\} \\ X_2 &= \{\{a, b, c\}\} \end{aligned}$$

and orderings $a < b < c$ and $c < b < a$ respectively (the strict $<$ simply indicates that a, b, c are in fact distinct), are the same object as abstract simplicial complexes.

Example 6.21. There is a functor

$$\mathcal{K} : \mathbf{Posets} \to \mathbf{OrdSCxs}$$

which takes the poset as the underlying poset for the ordered simplicial complex, and then takes all finite totally ordered subsets of the poset as simplices. Intuitively, $\mathcal{K}$ takes the "maximal" set of simplices described by the partial order.

6.4 Simplicial approximation

Definition 6.22. Let X be a topological space and K a simplicial complex. Continuous maps $f, g : X \to |K|$ are **simplicially close** if for all $x \in X$ we have $f(x), g(x)$ are both in the closure of some simplex $\sigma_x \subseteq |L|$.

Proposition 6.23. *If f, g are simplicially close, then $f \simeq g$.*

Proof. Take the homotopy

$$h : X \times I \to |K| \subseteq \mathbb{R}^N, \quad h(x,t) = tf(x) + (1-t)g(x).$$

By simplicial closeness, $f(x), g(x)$ lie in the closure of some simplex σ_x. By convexity, $h(x,t)$ is contained in σ_x which ensures continuity. (In the finite case, we can also observe $|K|$ has the subspace topology in $\mathbb{R}^N$.) □

Definition 6.24. A **subdivision** L of a simplicial complex K, written $L \subseteq K$, is where every simplex of L is contained in a simplex of K, and every simplex of K is a finite union of simplices of L. It follows that $|L| = |K|$.

Definition 6.25. The **barycenter** b_σ of a geometric simplex σ is given by $b_\sigma = \frac{1}{n+1}\sum_0^n(v_i)$. One constructs the **barycentric subdivision** K' of a simplicial complex K as the union $K = \bigcup L_n$, where the L_n are defined as follows, starting with $L_0 = K^0$:

- Let L_{n-1} be a subdivision of the $(n-1)$-skeleton K^{n-1}.
- For a n-simplex σ in K, take its boundary. This corresponds to an ordered subcomplex L_σ of L_{n-1}.
- We define

$$L_n = L_{n-1} \cup \bigcup_{\sigma \in K_n} b_\sigma * L_\sigma.$$

  where the $*$ is the **cone operation**, where for every simplex τ in L_σ we add an additional simplex $\tau \cup \{b_\sigma\}$.

In this way we get an ordered simplicial complex with vertices $\{b_\sigma\}$, where the ordering is given by $b_\sigma \leq b_\tau$ if $\sigma \subseteq \tau$.

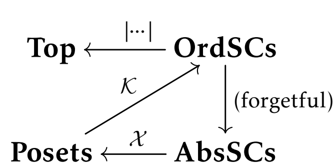
Definition 6.26. The barycentric subdivision procedure gives the functor

$$\mathcal{X} : \textbf{AbsSCxs} \to \textbf{Posets}$$

where K goes to the set of barycenters $\{b_\sigma\}$ with the partial order described above. The ordered simplicial complex K' is thus the result of the composition

$$\mathcal{K}\mathcal{X} : \textbf{AbsSCxs} \to \textbf{OrdSCxs}.$$

Example 6.27. To summarize, we have



where the *triangle does not commute*; following the arrows around from **AbsSCxs** will take you from a simplicial complex K to its barycentric subdivision K' (with the ordering forgotten).

Example 6.28. Consider a triangle K (as a simplicial complex). Constructing the barycentric subdivision proceeds as follows:

- We start with three points (the vertices) $L_0 = \{b_0, b_1, b_2\}$.
- For the original 1-simplices $\{b_i, b_j\}$, we take the barycenter (the midpoint), place a vertex b_{ij} there, and thus get L_1 by adding simplices $\{b_i, b_{ij}\}$, $\{b_{ij}, b_j\}$. Hence
$$L_1 = \{b_0, b_1, b_2, \{b_0, b_{01}\}, \{b_{01}, b_1\}, \{b_0, b_{02}\}, \{b_{02}, b_2\}, \{b_1, b_{12}\}, \{b_{12}, b_2\}\}$$
- For the original 2-simplex $\{b_0, b_1, b_2\}$, we take its barycenter (the centroid), place a vertex b_{012} there, which induces the creation of 6 new triangles (and associated edges) via the cone operation $b_{012} * L_1$ (since the boundary of the original 2-simplex gives $L_1 \subseteq L_1$ itself).

We are done after the 2-simplex, so $K' = L_2 = L_1 \cup (b_{012} * L_1)$. The barycentric subdivision K' has the natural partial ordering given by $b_{012} \geq b_{01}, b_{12}, b_{02}$ and $b_{ij} \geq b_i, b_j$. One can see that this partial order restricts to a total order on each simplex K'.

Definition 6.29. The **n-th barycentric subdivision,** $K^{(n)}$ of K is given by repeated barycentric subdivision; i.e., $K^{(1)} = K'$ and $K^{(i)} = (K^{(i-1)})'$.

Theorem 6.30 (Simplicial approximation theorem)**.** *For a continuous map $f : |K| \to |L|$ where K and L are simplicial complexes, there exists a number n and a simplicial map $g : K^{(n)} \to L$ such that $|g| \simeq f$.*

Proof. Not too difficult, but our entire treatment has been too informal to support it. See [Finite Book, §4]. □

The point is that if you have any continuous map between two spaces that are geometric realizations of (possibly infinite) simplicial complexes, the map is homotopic to a simplicial map (albeit now from a more complicated complex $K^{(n)}$). Simplicial maps are nice and piecewise, and we can now approximate many continuous maps with them.

Remark. Historically, the foundations of algebraic topology were established using simplicial complexes. Once we encounter CW complexes however, we will see that that they are an even more natural setting, especially for homotopy theory and its generalizations (due to Quillen; see Riehl's *Categorical Homotopy Theory*).

7 Chain complexes and simplicial homology

Remark. This section is based on Chapter 12 of *Concise*, from class, etc.

7.1 Simplicial chain complexes

Definition 7.1. A **chain complex** (M_*, ∂_*) over R is a sequence of R-modules M_i

$$\cdots \xrightarrow{\partial_{i+2}} M_{i+1} \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$

connected by **boundary operators** $\partial_i : M_i \to M_{i-1}$, which are homomorphisms that satisfy $\partial_i \circ \partial_{i+1} = 0$ for all i.

Notation 7.2. When indices are obvious, one might drop the indices. For example, the requirement on boundary operators is often simply expressed as $\partial^2 = 0$.

Example 7.3. The **chain complex** $C_*(K)$ **induced by an ordered simplicial complex** K is a chain complex over $\mathbb{Z}$, given by the sequence of free abelian groups on the n-simplices of K, i.e.,

$$C_n(K) = \left\{ \sum a_j \sigma_j \mid a_j \in \mathbb{Z}, \sigma_j \text{ is an } n\text{-simplex of } K \right\},$$

where the boundary maps are given by

$$\partial(\sigma) = \partial([x_0, \ldots, x_n]) = \sum_{i=0}^{n} (-1)^i [x_0, \ldots, \hat{x}_i, \ldots, x_n]$$

(where $\hat{x}_i$ means to remove the i-th component). One can verify that $\partial_i \circ \partial_{i+1} = 0$.

Example 7.4. Let K_n denote the set of n-simplices of K. Then in the chain complex $C_*(K)$, we have $C_n(K) = \mathbb{Z}[K_n]$. We can think of C_n as the **n-th chain complex functor**

$$C_n : \mathbf{AbsSCxs} \to \mathbf{Ab}, \quad K \mapsto \mathbb{Z}[K_n]$$

where the morphisms are transformed from $f : S \to T$ with

$$C_n f : \mathbb{Z}[S_n] \to \mathbb{Z}[T_n], \quad C_n f\left(\sum a_i s_i\right) = \sum a_i f(s_i).$$

7.2 Simplicial homology groups

Definition 7.5. The **n-th homology group** of a chain complex is given by

$$H_n := \ker(\partial_n)/\mathrm{im}(\partial_{n+1})$$

Definition 7.6. The **n-th simplicial homology group** (over $\mathbb{Z}$) of K, written $H_n(K;\mathbb{Z})$, is the n-th homology group of the induced chain complex $C_*(K)$.

Example 7.7. The n-th simplicial homology group is a functor

$$H_n : \mathbf{OrdSCxs} \to \mathbf{Ab}.$$

When we develop singular homology, we will see that this H_n factors through a functor $\mathbf{Top} \to \mathbf{Ab}$.

Exercise 7.8. *Consider the standard n-simplex Δ_n^s. Compute that $H_*(\Delta_n^s) = H_0(\Delta_n^s) \cong \mathbb{Z}$.*

Exercise 7.9. *$H_q(\partial\Delta_n^s; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ at $q = 0$ and $q = n-1$, and is 0 elsewise.*

Notation 7.10. One sometimes writes $Z_n := \ker(\partial_n)$ for the set of **cycles** and write $B_n := \mathrm{im}(\partial_{n+1})$ for the set of **boundaries**. This leads to the phrase that "homology is cycles mod boundaries." The terminology is inspired by the geometric intuition given by simplicial homology.

7.3 Maps of chain complexes

Definition 7.11. A **map of chain complexes** $f : (M_*, \partial_*) \to (M'_*, \partial'_*)$ is a sequence of maps $f_i : M_* \to M'_*$ such that the following diagram commutes for all i:

$$\begin{array}{ccc} M_i & \xrightarrow{f_i} & M'_i \\ \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \partial'_i} \\ M_{i-1} & \xrightarrow{f_{i-1}} & M'_{i-1} \end{array}$$

That is, we must have $f_{i-1} \circ \partial_i = \partial'_i \circ f_i$.

Example 7.12. We have the inclusion of simplicial complexes

$$\partial\Delta_n^s \to \Delta_n^s.$$

This induces a map of (simplicial) chain complexes ι which is the identity for $i < n$. Consider what changes: in the Δ_n^s case, we now have a single n-simplex. We know that the space Δ_n^s is contractible, so it has the homology of a point (0 everywhere except for $H_0 \cong \mathbb{Z}$). However, if $C_n(\Delta_n^s) \cong \mathbb{Z}$ and $C_n(\partial\Delta_n^s) = 0$ as observed, then $\mathrm{im}(\partial_n)$ is empty (vs. being $\mathbb{Z}$) and hence $H_i(\partial\Delta_n^s) \cong \mathbb{Z}$ for $i = 0, n-1$, while remaining 0 elsewhere.

Exercise 7.13. *Work out all the details of the inclusion of the boundary of the 2-simplex into the 2-simplex.*

Definition 7.14. The **induced map on homology** f_* is the set of maps

$$f_{*i} : H_i(M_*, \partial_*) \to H_i(M'_*, \partial'_*),$$

where the notation $H_i(\cdots)$ here refers to the i-th (abstract) homology group induced by the respective chain complex, and where f_{*i} is defined in the obvious way (on equivalence classes in $\ker(\partial_n)/\mathrm{im}(\partial_{n+1})$; one verifies the map is well-defined).

Remark. When do two maps of chain complexes induce the same map on homology (groups)? The answer to ths will turn out to be the notion of homotopy between maps of chain complexes. This is analogous to homotopic maps of spaces inducing the same map on homotopy (groups)!

Definition 7.15. A **homotopy between maps of chain complexes** $s : f \simeq g$ where $f, g : (C_*, \partial_*) \to (C'_*, \partial'_*)$, consists of homomorphisms of abelian groups $s_n : C_n \to C'_{n+1}$ that satisfy

$$\partial' \circ s + s \circ \partial = f - g$$

for all indices. This has the property that if $f \simeq g$, then $f_* = g_* : H_*(C_*) \to H_*(C'_*)$. (intuitively, $f(x) - g(x)$ "represent the same boundary" when $(ds + sd)(x)$; see https://en.wikipedia.org/wiki/Homotopy_category_of_chain_complexes for explanation and a diagram).

We will now describe an example of homotopic maps of chain complexes via the cone construction.

Definition 7.16. Given an ordered simplicial complex K with geometric realization X (via passage through **SCxs**), we define the **cone** CX as the quotient

$$CX = (X \times I)/(X \times \{1\})$$

(imagine taking the cylinder $X \times I$ and then identifying one of the ends to be a single point). The **cone of an ordered simplicial complex** is the ordered simplicial complex CK (with $|CK| = X$) is given by taking:

- The vertex set is $V(K)$ plus the cone vertex x which is greater than all elements of $V(K)$
- For every n-simplex σ, we also create an $(n+1)$-simplex given by $\sigma \cup \{x\}$.

Example 7.17. $\Delta^s_n \cong C\Delta^s_{n-1}$.

Exercise 7.18. *The cone is a contractible space (homotopy equivalent to a point).*

Example 7.19. We have $C_n(CK)$ as the free abelian group with basis

$$K_n \sqcup \{\{x\} \cup \sigma \mid \sigma \in K_{n-1}\}.$$

We will show that $\mathrm{id}, \epsilon : C_*(CK) \to C_*(CK)$ are chain homotopic maps, where

$$\begin{aligned} \epsilon_n : &\quad C_n(CK) \to 0 \text{ for } n \neq 0 \\ \epsilon_0 : &\quad C_0(CK) \to \mathbb{Z} \text{ (generated by } \epsilon_0(x)) \end{aligned}$$

Our chain homotopy is given by

$$s_n : C_n(CK) \to C_{n+1}(CK), \quad s(\sigma) = (x, \sigma),\ s(x, \tau) = 0,$$

(i.e., if a simplex contains the cone vertex, it goes to 0; if it does not, take it to the simplex with the cone vertex adjoined).

Now observe that

$$\partial s(\sigma) = \partial(x, \sigma) = \sigma - (x, \partial\sigma), \quad s\partial(\sigma) = (x, \partial\sigma), \quad \partial s(x) + s\partial(x) = 0 + 0 = 0.$$

Altogether, we conclude

$$\partial s + s\partial = \mathrm{id} - \epsilon.$$

where the first two observations deal with the case $n > 0$, and the last observation deals with $n = 0$.

7.4 Tensor products of chain complexes

Definition 7.20. The **tensor product of chain complexes** $(M_*, \partial_*), (N_*, \partial'_*)$ over R is the chain complex $(M_* \otimes N_*, d_*)$ given by the sequence of R-modules

$$(M_* \otimes_R N_*)_n = \bigoplus_{i+j=n} M_i \otimes_R N_j,$$

where the n-th differential is

$$d(x \otimes y) = \partial(x) \otimes y + (-1)^i x \otimes \partial'(y).$$

for $x \in M_i$, $y \in N_j$.

Exercise 7.21. *Verify that $d^2 = 0$.*

Using tensor products, we can motivate the earlier definition of homotopy for chain complex maps (namely, the $ds + sd = f - g$ requirement) by analogy to the definition of homotopy for topological spaces.

Example 7.22. Consider $I = \Delta_1^s$, which we will also use to denote its chain complex. Let s be a chain homotopy between maps $f, g : M_* \to N_*$. There is a chain map

$$h : M_* \otimes I \to N_*.$$

such that $h(x \otimes [1]) = f(x)$ and $h(x \otimes [0]) = g(x)$.

Exercise 7.23. *Let $s(x) = (-1)^{\deg x} h(x \otimes [1])$. Show the existence of h is equivalent to the $ds + sd = f - g$ requirement.*

Remark. For simplicial (and CW complexes), there is an isomorphism $C_*(X \times Y) \cong C_*(X) \cong C_*(Y)$. Then we can view

$$C_* : \mathbf{OrdSCxs} \to \mathbf{ChCxs}$$

as a functor that preserves the notion of homotopy in the respective categories, via

$$h : X \times I \to Y \quad \to \quad C_*h : C_*(X \times I) = C_*(X) \otimes C_*(I) \to C_*(Y).$$

8 [TODO] Simplicial objects

Remark. Content + exercises + etc. for this is scattered across lectures, heaven't organized them. Summary for now, see [Finite, §10] or May's *Simplicial Objects* for further details.

8.1 Summary

- Degenerate complexes K^+ (simplices can have repeated vertices); simplicial complexes as special case of degenerate complexes (which we can think of as **simplicial sets**)
- Can define differentials and thus $C_*^+(K)$ complexes; this time, C_*^+ is itself a functor
- K and K^+ give same homology: `https://math.uchicago.edu/~may/REU2016/Normalized.pdf`
- Defined functor from spaces to simplicial sets
- We now have

$$\mathbf{Spaces} \to \mathbf{SimplSets} \to \mathbf{SimplAb} \to \mathbf{ChComp} \to \mathbf{GradedAbGroups}$$

- We also have **SimplSets** $\to$ **Spaces** by geometric realization.
- (In later lectures: introduced the simplex category and (implicitly) **simplicial objects**

9 CW complexes

Remark. The following treatment is largely from class, which May admits is a sampling from *Concise Course*. Some of the definitional exposition is also from [Hatcher, Chapter 0].

9.1 Motivation and definition

The concept of the CW complex was invented by J.H.C. Whitehead. As spaces, they are more general than simplicial complexes, while still having the nice property that weak equivalence of CW complexes is homotopy equivalence (Whitehead's theorem). Furthermore, every space is weakly equivalent to a CW complex (CW approximation theorem).

As to what a CW complex is, they are products of a process that is somewhat reverse to giving a space a simplicial decomposition. Consider the simplicial complex described by a tetrahedron. Instead, imagine starting with the 0-simplices (the vertices), between which we introduce 1-simplices (the edges), along which we introduce 2-simplices (the faces). In a sense, CW complexes construct a space from the bottom-up, as opposed to how simplicial complexes were induced by some top-down "saturation".

Definition 9.1. A **CW complex** is space X that can be constructed in the following manner: Let X^0 be a set with the discrete topology (we call its elements the **vertices**). The space X is then

$$X = \bigcup_{n=0}^{\infty} X^n$$

where the sequence $X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n$ is defined inductively as follows:

- Given X^{n-1}, take a set of **attaching maps** (possibly empty)

$$j_\alpha : S^{n-1} \to X^{n-1}.$$

- Along each of these maps, we attach a copy of D^n (identifying the boundary of each D^n with each image $j_\alpha(S^{n-1})$).

- We define the n-**skeleton** X^n to be the pushout in the following diagram.

$$\begin{array}{ccc} \bigsqcup_\alpha S^{n-1} & \xrightarrow{\bigsqcup_\alpha j_\alpha} & X^{n-1} \\ \big\downarrow & & \big\downarrow \\ \bigsqcup_\alpha D^n & \longrightarrow & X^n \end{array}$$

- That is, it is the quotient space given by the disjoint union $X^{n-1} \sqcup \bigsqcup_\alpha D^n$ under the identification $x \sim j_\alpha(x)$ for every $x \in \partial D^n_\alpha$. Intuitively, the boundary of the α-th ball is identified with the image of the α-th sphere in X^{n-1}.

- The n-**cells** of X are ultimately the images of each $D^n \to X^n \subseteq X$.

The process may stop at a finite stage (i.e., there are no more attaching maps after some $m < \infty$ and thus $X = X^m$). The largest n for which are n-cells is the **dimension** of the CW complex. Otherwise, things might continue indefinitely where the union is given the weak topology (i.e., a set U is open if and only if $U \cap X^n$ is open for all n).

Example 9.2. Topologically, **graphs** are exactly 1-dimensional CW complexes (when we say graph, multiple edges between vertices and loops at a vertex are both allowed). The points on the graph are X^0, and the edges are intervals (copies of D^1) glued along pairs of vertices in X^0 (i.e., images of S^0) to give $X = X^1$.

Example 9.3. The n-sphere S^n can be viewed as a CW complex with one 0-cell and one n-cell. Take $X^0 = \{\star\}$, then $X^{n-1} = \cdots = X^0$, then a single attaching map $j : S^{n-1} \to X^{n-1}$ as the constant map to $\star$ to give

$$X^n = \{\star\} \sqcup D^n/\sim \cong D^n/\partial D^n,$$

since $\sim$ identifies $\partial D^n \sim \star$ and thus every boundary point to each other. Since $S^n \cong D^n/\partial D^n$, we conclude that $S^n \cong X = X^n$.

Example 9.4. **(TODO: describe the real projective space CW complex)** The point is that a simplicial triangulation of $\mathbb{R}P^n$ is annoying (see https://mathoverflow.net/questions/50382/how-to-triangulate-real-projective-spaces-as-simplicial-complexes-in-mathematic) , while our CW complex construction is rather nice.

9.2 Operations on CW complexes

Example 9.5. Geometric realizations of simplicial complexes are a specialization of CW complexes:

$$\begin{array}{ccc} \partial\Delta_n^t & \longrightarrow & K^{n-1} \\ \downarrow & & \downarrow \\ \Delta_n^t & \longrightarrow & K^n \end{array}$$

Definition 9.6. The **product of CW complexes** X and Y is given where the n-skeleton is

$$(X \times Y)^n = \bigcup_{p+q=n} X^p \times Y^q.$$

We have

$$\begin{gathered} D_j^p \times S_k^q \cup S_j^{p-1} \times D_b^q \subseteq D^p \times D^q \\ I^p \times \partial I^q \cup \partial I^{p-1} \times I^q \subseteq I^p \times I^q \end{gathered}$$

where I is the CW complex with vertices 0, 1 and a 1-cell glued to them.

Definition 9.7. A **cellular map** between CW complexes is a map that takes n-skeletons to n-skeletons; i.e., $f : X \to Y$ satisfies $f(X^n) \subseteq Y^n$ for all n.

Definition 9.8. A **cellular homotopy** is a homotopy $h : X \times I \to Y$ which is a cellular map.

9.3 [TODO] Cellular approximation, etc.

Theorem 9.9 (Cellular approximation)**.** ***(TODO: haven't done the proof yet, just alluded to it; requires notion of relative CW complexes, not covered until July 21)***

Proof. **(TODO: the homotopy extension lifting lemma (HELP) comes in that allows homotopies to be done cell by cell, then inductively on dimension.)** □

Corollary 9.10. *Any $f : X \to Y$ is homotopic to a cellular map. If f is cellular, it induces a map $C_\#(X) \to C_\#(Y)$ of cellular chain complexes. If $h : f \simeq g$ is any homotopy where f, g are cellular, then $f \simeq g$ via a cellular homotopy.*

Remark. Intuitively, it suffices to consider cellular maps to study the up-to-homotopy space $[X, Y]$.

(TODO: lots of other stuff was said, left out for now)

Part II
Week 4 onwards

Remark. I do have actual day-by-day notes on these, but they're half TODOs and things I didn't understand. Leaving them out for now and just summarizing what was discussed. Will get to this by the time Alg. Top. lectures resume (if not, then end of August; if not, then probably never).

10 Topics covered

Remark. References:

- May's *Concise Course*: https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf
- May's *Finite Book*: https://math.uchicago.edu/~may/FINITE/FINITEBOOK/FINITEBOOKCollatedDraft.pdf

10.1 July 11 (May)

- Quotienting by contractible space (e.g., maximal tree of graph) preserves homology
- Defined cofibrations and fibrations, and the corresponding homotopy extension property (HEP) and homotopy lifting property (HLP). Mapping cylinder and co-cylinders. See [Concise, §6-7].
- We also defined pullbacks (see Categorical Notions section).
- Fibrations generalize unique path lifting of covering spaces.
- Constructed a map that is 0 on homology/homotopy but not nullhomotopic (this one: http://mathoverflow.net/a/20303)

10.2 July 12 (May)

- Idea of replacing arbitrary maps by cofibration/fibration up to homotopy
- Neighborhood deformation retracts (NDR) [Concise, §6]
- How to recognize fibrations
- The long exact sequence for cofibrations [Concise, §8.4]

10.3 July 13 (May)

- The dual long exact sequence for fibrations [Concise, §8.5]
- The homotopy group long exact sequence [Concise, §9]
- Cohomology, take 1 (via cochains and cochain complexes)
- "Homology is useless"; cohomology has a graded ring structure
- Cohomology, take 2 (via Eilenberg-MacLane spaces)
- "Deduced" the Eilenberg-Steenrod axioms
- Generalized cohomology theories as those satisfying all but dimension axioms

10.4 July 14 (Chan)

- The simplex category
- Geometric realization of a simplicial object
- The 2-sided bar construction (https://ncatlab.org/nlab/show/two-sided+bar+construction)
- Classifying spaces of categories via functor B (or posets, or groups)
- Properties of B:

  a) B is functorial (i.e., $f : G \to H$ induces a map $Bf : BG \to BH$). ((**TODO: offhand comment about homotopy invariance of B with respect to G-equivariance**), e.g.,

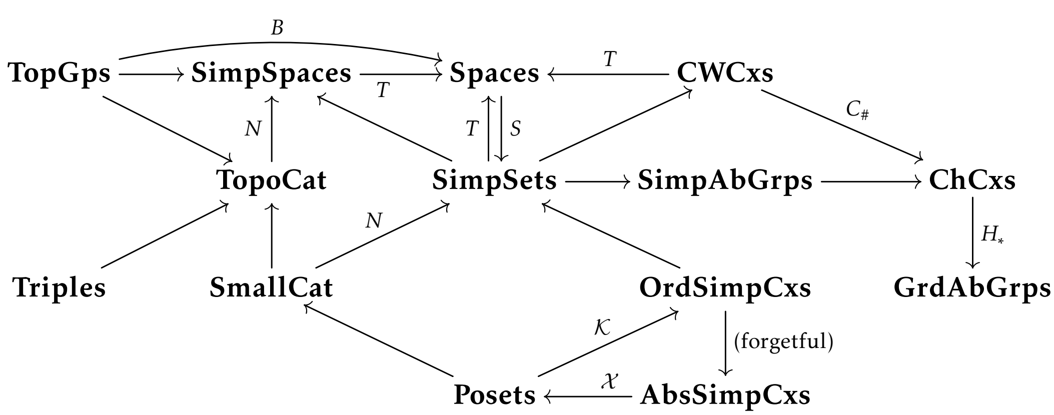
  $$\simeq : * \times G^n \times * \to * \times H^n \times *)$$

  b) EG is contractible
  c) $G \simeq \Omega BG$ (if G is good)
  d) $EG \to BG$ is the universal principle G-bundle
  e) $B(G \times H) \cong BG \times BH$
  f) If G is abelian then BG is a group

- (Proved or at least discussed these by various means)
- One of those proofs used the five lemma from homological algebra (https://en.wikipedia.org/wiki/Five_lemma)

10.5 July 15 (May)

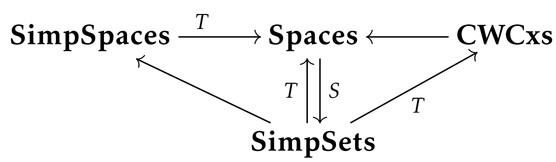
- Pretty much this diagram:



- The nerve functor N and its properties
- The category **Triples**
- Idea: translate differential topology problems in terms of vector bundles, and then use homotopy theory (via classifying spaces)
- Defined adjoint functors (see Categorical notions)
- Diagram should commute everywhere? (except barycentric triangle at the bottom?)

10.6 July 18 (May)

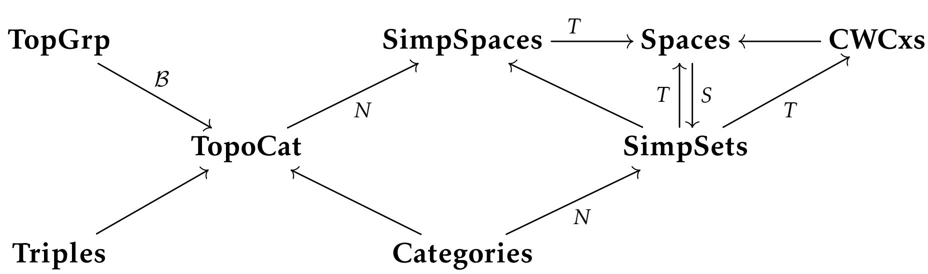
- Focused on this part:



- Discussed the S, T maps as adjoint pairs of functors
- Mentioned Whitehead's theorem
- Described the T functor on simplicial sets in detail, e.g., [Finite Book, §10.6]
- Singular chain complexes and cell chain complexes as equivalent

10.7 July 19 (May)

- Diagram for the day



- Initial and terminal objects
- Classifying space of a category $B\mathcal{C} = |N\mathcal{C}|$

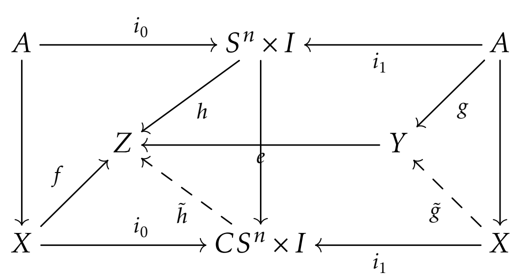
- Topological categories as categories whose hom-sets are topological spaces
- $\mathcal{E}$ as a related functor to $\mathcal{B}$
- $p : EG \to BG$ as universal principal bundle; every G principal bundle over paracompact space is a pullback of this: https://en.wikipedia.org/wiki/Classifying_space

10.8 July 20 (May)

- Nerve and T functors preserve products for nice spaces
- Simplices as increasing sums (https://en.wikipedia.org/wiki/Simplex#Increasing_coordinates)
- Proved that S, T are homo topy preserving functors
- Idea: do all of homotopy theory in simplicial sets
- By passing to chains, identical homotopy induces identical homology; done without appeal to topology!
- Constructed the chaotic category (https://ncatlab.org/nlab/show/indiscrete+category) for X: $\tilde{X}$. Turns out $\tilde{G} \cong \mathcal{E}G$

10.9 July 21 (May)

- n-equivalences
- Relative CW complexes [Concise, §10.1]
- HELP and Whitehead theorem [Concise, §10.3]



- Power of Whitehead's theorem (and CW complexes): n-equivalence of CW complexes with dimension less than n is homotopy equivalence

10.10 July 22 (May)

- Used HELP to (sketch) proof of **cellular approximation**; see [Concise, §10.4]
- **Excisive triads**, and the "original" axioms of homology: exactness, excision (see **Eilenberg-Steenrod Axioms**)
- Used A as upper-2/3s, B as lower-2/3s, in the excision axiom to see how suspension shifts indices of homology by 1
- Introduced additivity, weak equivalence axioms
- (The above treatment is [Concise, §13])
- We then did based case and **reduced homology**; excision is meaningless, so we use **suspension axiom** [Concise, §14]
- Technicalities about CW triads vs excisive triads; introduced **homotopy pushouts** (pushouts where the square commutes only up to homotopy) [Concise, §10.7]